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T³ EUROPE

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Modelling Increase in Rainfall with Temperature

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1. Learning objectives and system aspects

a. Learning outcomes

At the end of this exercise students are

- Able to enter a complicated function into the graph page.
- Able to use the function notation in calculations.
- Know the difference between exponential and non-exponential change.
- Know that global warming will lead to increased rainfall.
- Understand how mathematical models support scientific models.

b. Science background

Thermal energy from the Sun evaporates water from the oceans. The amount of energy needed is known as the Latent Heat of evaporation because the change in phase from water to water-vapour takes place at constant temperature. The two properties of interest are the water-vapour pressure P and the temperature at which the phase change is taking place T. The water-vapour pressure determines how much water is in the atmosphere and consequently how much rain can fall. The mean global temperature is the temperature at which the phase change is taking place and this mean global temperature is slowly rising due to climate change.

The relationship between P and T was developed by Clapeyron (sometimes known as the Clausius-Clapeyron equation) around 1840.

$$\frac{dP}{dT} = \frac{L}{T\Delta V}$$

The solution to the Clapeyron equation by August-Roche-Magnus is used today to model how the rainfall will change with mean global temperature.

$$P(\theta) = 0.61e^{\left(\frac{18\theta}{\theta+243}\right)}$$

Where θ is the temperature in °C.

The Clapeyron equation is a triumph of thermodynamics which is the science of the behaviour of gases liquids and solids as energy is transferred between them. The oceans and atmosphere represent an enormous thermodynamical system and climate models all use thermodynamics to model the atmosphere and predict the future!

One of these predictions concerns the increase in rainfall as the mean global temperature rises. The atmosphere is able to hold more water vapour as the temperature rises and this results in ever greater amounts of rain. This extra rainfall is not distributed uniformly over the surface of the Earth, some regions experience more, while others less.

c. Connection with Sustainability Development Goals

There are 17 SDGs. (https://sdgs.un.org/goals).

Here sustainable development goals 6 ‘Clean Water and Sanitation’ and 13 ‘Climate Action’ are addressed.

Clearly without rain fresh water is difficult to access, but too much results in catastrophic flooding such as has been witnessed in various parts of the world. At the same time some regions are experiencing drought.

There are a number of ways climate change may contribute to drought. Warmer temperatures can enhance evaporation from soil, making periods with low precipitation drier than they would be in cooler conditions. Droughts can persist through a “positive feedback,” where very dry soils and diminished plant cover can further suppress rainfall in an already dry area.

2. Apparatus needed

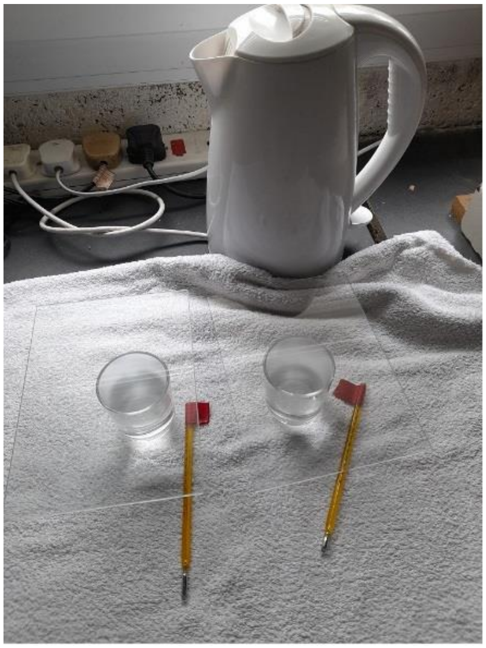
Two small 100 cc beakers

Two thermometers 0-100 C.

Kettle for hot water

Two glass or plastic sheets to cover the beakers.

Stop watch



3. Experiment setup

Arrange the beakers so that they are about half full with water at two different temperatures 15 C° apart. This will ensure that the vapour pressure at the surface of the water in the warmer beaker is about double that of the colder one and will further ensure that condensation will take place for both glass plates when placed on top of the beaker.

Since P is double it follows that the rate of condensation will also double hence we would expect to see the same amount of condensation for the colder beaker as the warmer one if the colder beaker is given four times the time.

Remove the thermometers and place the glass plate over the colder beaker and time for 15 seconds. Immediately cover the warmer beaker and time for a further five seconds. This ensures that the colder beaker has had 20 seconds and the warmer beaker just 5 seconds. Hold the plates up and compare the amount of condensation on each.

It is worth pointing out to students that water vapour is an invisible gas and that steam is condensed water vapour. It is not obvious to many students water is continually evaporating from bodies of water such as puddles, ponds, lakes seas and oceans. Water vapour in itself is a gas which is responsible for the greenhouse effect and if it is increased then an enhanced greenhouse will occur leading to climate change.

Since the experimental work verifies the mathematical model (albeit crudely) students should work through the “rainfall model.tns” file first

Use the rainfall model.tns file for students and allow them to work through the windows, entering the mathematical model and exploring the effect on P of an increasing global temperature.

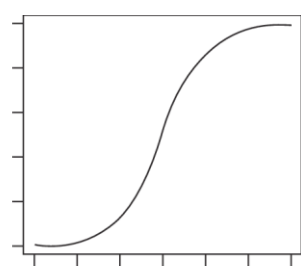
4. Activity and Tips

Models suggest that rainfall will increase by 7% for every degree rise in mean global temperature. Students will use the mathematical model employed by climate change scientists to verify this statement. The tasks will allow them to practise entering a function correctly, practise percentages and practise the use of function notation.

Setting x to zero in the denominator of the exponent yields an exponentially increasing function. The denominator x appears because the Latent Heat of evaporation itself changes with temperature.

Students will have difficulty entering the exponential (if not encountered before). It might be useful to start off the activity by asking students to enter exponentials and explore this first.

Strict exponential increase or decrease implies a constant doubling or halving time. In reality there are many factors which affect natural processes and so natural “exponential” changes tend to have inflexion points where increased growth stops and decreasing growth begins. This will continue until growth stops and the phenomenon being observed (Covid19, bacteria in a petri plate, lily pads on a pond) and the resulting curve will be a logistic curve as shown.



The short, very simple, very crude piece of practical work should be done after the mathematical model is explored. Students can use function notation to determine the difference in temperature for a doubling of P, or simply note that it is about 15 C°. I find that a range between 40 °C – 55 °C and 50 °C – 65 °C works well and avoids the use of very hot water.

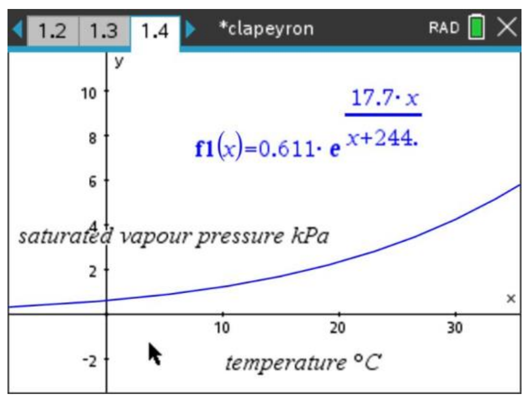
For more advanced students the Clapeyron equation derivation sheet could be used with appropriate sections missing to stimulate a response. The solution provided uses only concepts used by students at 17 or 18 years old and in principle could be followed, although the derivation and solution is fairly long. Calculation of the constant c after the integration would, for example, be an appropriate task.

In the experiment described the analysis is based purely on observation. You could ask students to devise a method to measure the amount of condensation on each plate.

5. Exemplar results

$P(\theta) = 0.61e^{\left(\frac{18\theta}{\theta+243}\right)}$
where P is the saturated vapour pressure and θ is the temperature in °C.

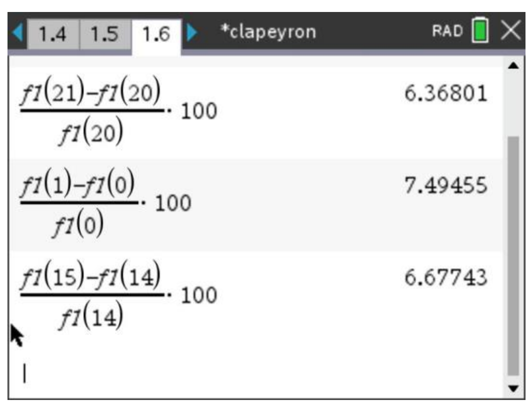
Enter the equation into the following graph page.



Use the following calculator page to estimate the percentage increase in P from 14°C to 15°C.
Is the increase the same for other 1°C rises?
Why is the increase in P not exponential?

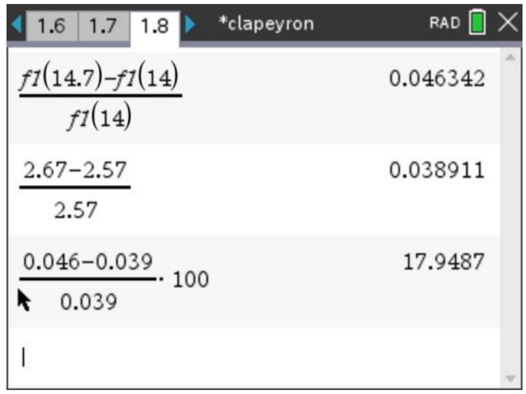
Equal increments in the domain do not yield equal increments in the range.
Ratios of values in the range for equal domain increments are not constant.

Principally because of the x (or T) in the denominator of the exponent.



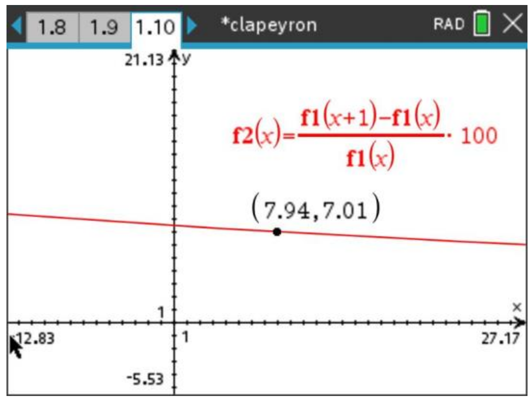
On the following page estimate what the model predicts for the increase and compare it with reality.
What is the percentage error associated with the model?

An 18 percent error is quite large which is why it is only necessary to quote values to 2 (or 3) significant figures.
Perhaps the model could be improved, or perhaps the data are inaccurate.



Construct a function which shows how the percentage increase in precipitation is decreasing with temperature.
Estimate the mean global temperature when rainfall was increasing at 7%/°C.

It is evident from the graph that the increase in rainfall with temperature is fairly constant (ie nearly exponentially increasing) over the ranges which concern us.



The energy required to change the temperature of the atmosphere and the oceans is enormous. Small changes of temperature represent vast amounts of energy (in or out) and this energy is not evenly distributed across the globe.
The result is that different geographic regions will be impacted by disproportionate weather events, some of which will be catastrophic as far as life on Earth is concerned.
The final question on the student sheet is about the lag time between the heating of the atmosphere and the heating of the oceans. The heat capacity of water is much greater than that of air and so takes longer to warm up. Predicting the final equilibrium temperature of planet Earth is the subject of other models.

6. Refs and extension material

a. Derivation of the Clapeyron equation

Water vapour evaporates from the oceans and condenses to rain, thus returning to the oceans and completing the water cycle. This is therefore a reversible isothermal process and so the change in internal energy, ΔU, of the water as it becomes vapour is zero. Evaporation occurs at constant temperature and pressure, but the vapour pressure, *P* , itself increases with temperature so that at higher temperatures the atmosphere holds more water, so consequently rainfall will increase.

[*Note that during a phase change there are changes in entropy and volume. The following derivation of a relation between P and T during the phase change does not reference the entropy change, ΔS which is accommodated in the ΔQ of the thermal flow, $\Delta Q = T\Delta S$.*]

The First Law of Thermodynamics states that the change in internal energy of a system is equal to the thermal energy flowing in minus the work done by the system, $P\Delta V$.

$$\Delta U = \Delta Q - P\Delta V$$

The thermal energy flowing into one mole of water at the phase transition temperature is simply the Latent Heat of vaporisation, L. ΔU is zero so therefore

$$0 = L - P\Delta V \quad \ldots\ldots\ldots 1$$

For 1 mole of an ideal gas we have

$$PV = RT$$

Therefore

$$P = \frac{RT}{V} \quad \ldots\ldots\ldots 2$$

Substituting 2 into equation 1 we get

$$L = \left(\frac{RT}{V}\right)\Delta V$$

For a particular volume of gas, $P = f(T)$ at the phase change, so from equation 2 we have

$$\frac{dP}{dT} = \frac{R}{V}$$

Therefore

$$L = \left(\frac{dP}{dT}\right)T\Delta V$$

Which rewritten becomes the Clapeyron equation:

$$\frac{dP}{dT} = \frac{L}{T\Delta V}$$

Solving the Clapeyron equation

The volume change, ΔV, during the evaporation of a mole of water is very large, so that the change is effectively equal to the final volume V if the initial volume is small,

$$\frac{L}{T} = \left(\frac{dP}{dT}\right) V$$

And so eliminating V

$$\frac{L}{T} = \left(\frac{dP}{dT}\right) RT/P$$

Rewriting, separating differentials and integrating

$$\int \frac{dP}{P} = \int \left(\frac{L}{RT^2}\right) dT$$

$$lnP = -\frac{L}{RT} + c$$

Calculating c

P is 0.61 kPa at 273 K and L is $4.5\text{x}10^4$ J per mole. R is 8.3 J/mole/K

$$c = ln0.61 + \ 4.5\text{x}10^4/(8.3\text{x}273)$$

Therefore

$$lnP = -\frac{L}{RT} + ln0.61 + 20$$

$$ln\frac{P}{0.61} = \ -\frac{L}{RT} + 20$$

Unfortunately *L* varies with temperature so that

L $=4.5\text{x}10^4$ -38θ (to the first order) where θ is in Celsius degrees, and *P* is in kPa.

$$P = 0.61\ e^{\left(-\frac{4.5\times10^4-38\theta}{R(\theta+273)}+20\right)}$$

$$P = 0.61\ e^{\left(\frac{15\theta}{\theta+273}\right)}$$

The solution above is derived using an empirical formula for L in terms of θ which may vary slightly depending on source. In reality this formula has terms in θ^2 , θ^3 and so on, but these higher order terms have been ignored here. In the literature one can find the August-Roche-Magnus solution quoted as follows,

$$P = 0.61\, e^{\left(\frac{18\theta}{\theta+243}\right)}$$

This equation shows that global rainfall will increase by approximately 7% for every degree rise in mean global temperature.

b. References

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