The use of CAS in exams

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A lecture at the T3 conference in Chicago, March 2017

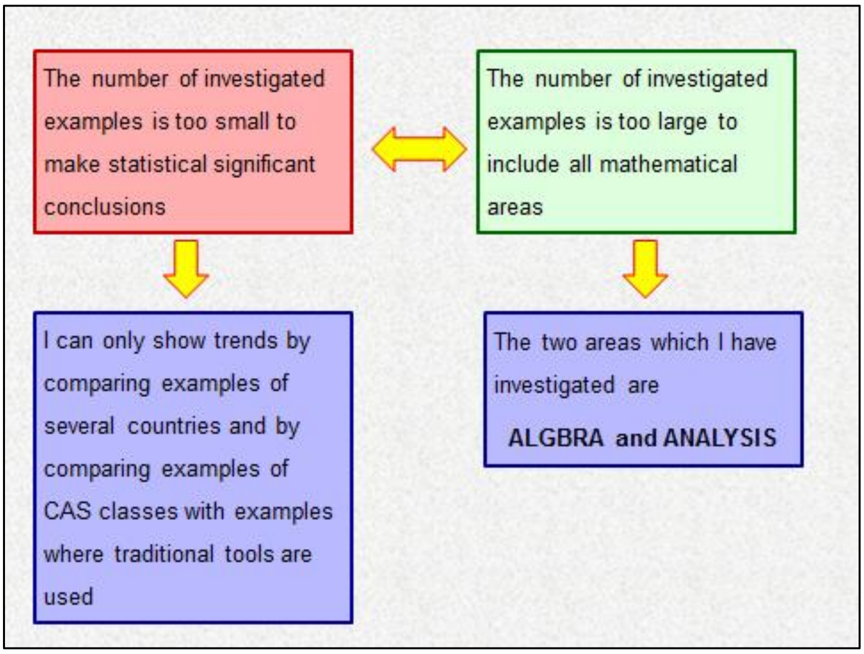
In the first part of the lecture, frameworks like the NCTM Standards and German or Austrian Competence Models for Standards will be presented which are necessary for the following studies.

The lecture will continue by investigating the influence of CAS in the quality and structure of exams (written exams in the classroom as well as final exams). Exams in countries where CAS tools are admitted will be compared with exams in countries where CAS is forbidden. Examples of exams in countries like Norway, Denmark, Germany, Austria, Australia will be compared.

Generally the influence of tests in the quality and the output of the educational process will be discussed and changes caused by the use of CAS in teaching, learning and testing will be demonstrated.

I have investigated final exams of 5 countries: Australia, Denmark, Germany, Norway and Austria.

What can such an investigation achieve and what not?



I am not so much interested in exams but I agree with A. Pallack who said: "*What is not tested will not be taught and learned.*"

It is impossible to improve the quality of mathematics education without improving the quality of exams. Especially in Austria we can observe an intensive "teaching to the test" caused by a new final exam. My dream was a "testing to the teaching", I wanted to develop an exam which has a positive influence on the quality of mathematics education.

1. The theoretical background, a framework for the investigation

Before presenting concrete examples it is necessary to describe a framework of the investigation. When studying the quality of exams following questions should be considered:

(1) Why mathematics? What is the goal of mathematics education?
(2) What competences are taught? – What competences are examined?
(3) How important is CAS for solving problems in exams?

Question 1: Why mathematics?

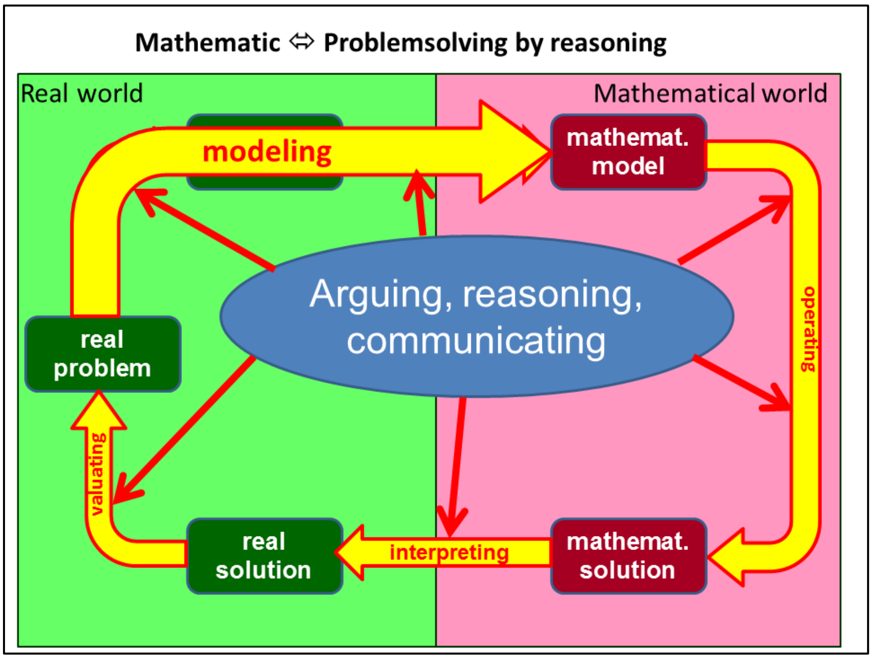
Not until we have considered the contribution of mathematics to a general education we should think about the role of technology for a better mathematics education.

It was very interesting not only to analyze exams of several countries but to look at their curricula and to study their answers to the question "why mathematics". To describe all the answers would be an extra lecture. I think Bruno Buchberger has formulated a summary with his definition of mathematics:

- *Mathematics is the technique of problem solving by reasoning, refined throughout the centuries*

I also agree with his second thesis:

- *Mathematical thinking technology is the essence of science and the essence of a technology based society*



The problem solving process often starts with real problems with data, with verbal information. This information has to be condensed and structured to come to a so called "real model" which can be translated into the language of mathematics to get a mathematical model.
This part of the process which leads to a mathematical model we call **modeling.**

By **operating** a mathematical solution should be found. **Interpreting** leads to a real solution which has to be evaluated to decide the solution of the problem. **Arguing and reasoning** accompanies the whole process.

Question 2: What does „competence“ mean? What mathematical competences should be taught“

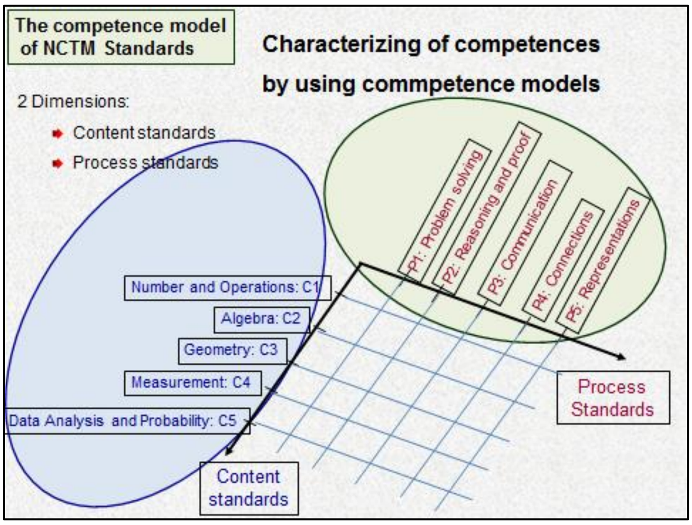
I try to make a clarification of terms: Competence versus skills

Competence *is the sustained available cognitive ability to solve problems in variable situations and to use the results responsibly united with the motivational, volitional and social willingness.* [Weinert]

A skill *to do something is the ability to carry out a certain activity like calculating with numbers or variables.*

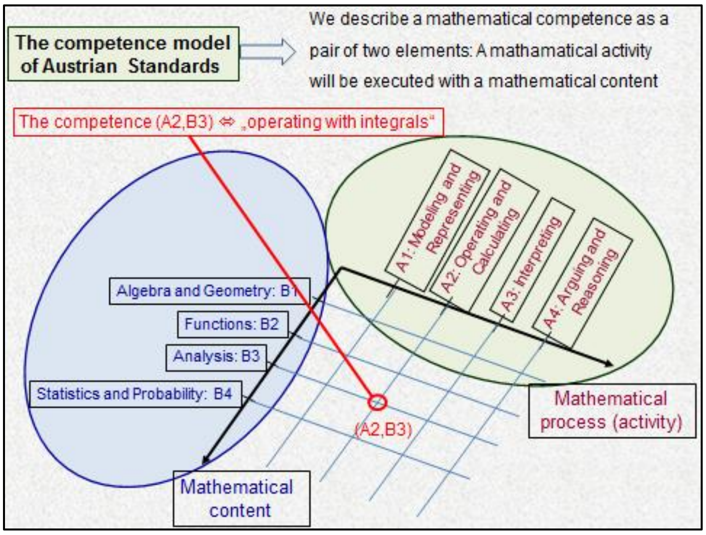
A mathematical competence *can be characterized by describing a mathematical activity executed with a mathematical content.*

A framework for analyzing mathematical activities are competence models:



The most famous competence model is the NCTM Standard model. Competences are describe by two dimensions :

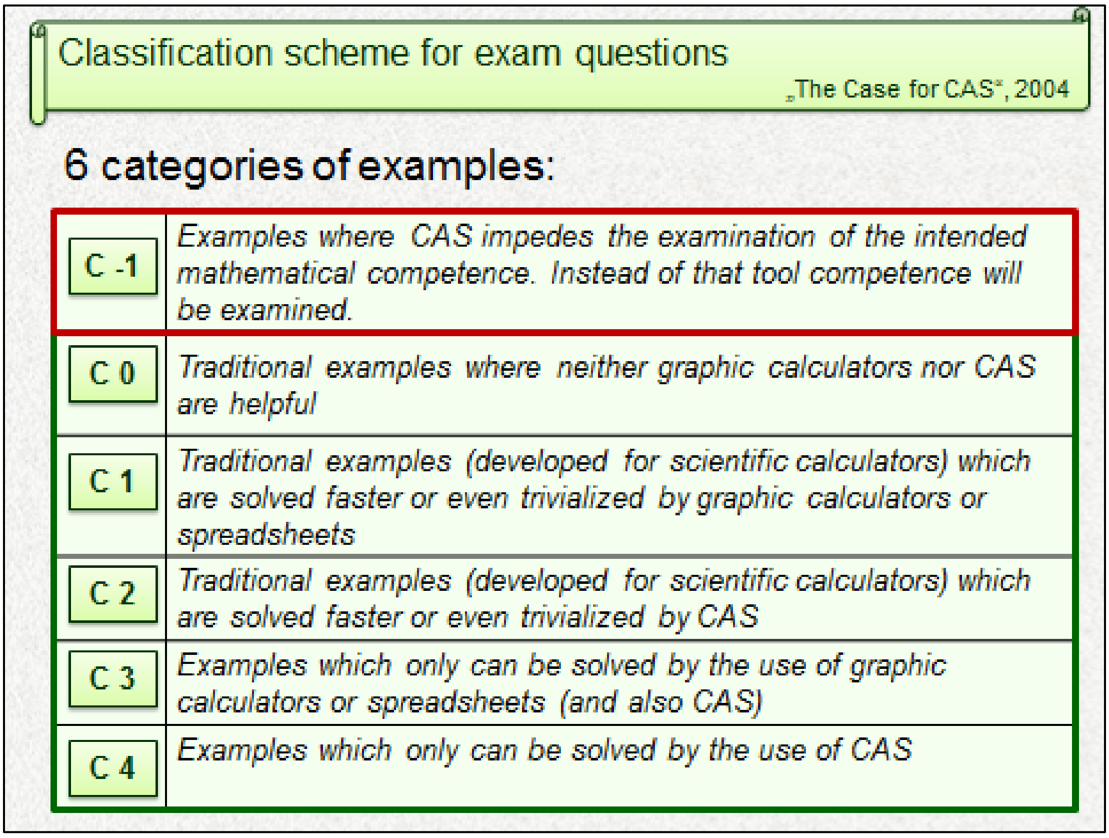
- Process standards are describing the mathematical activities,
- Content standards characterize the mathematical contents



In Austria we describe a mathematical competence as a pair of two elements: A mathematical activity (modeling, operating, a.s,o.) will be executed by a mathematical content (Algebra, Analysis, a.s.o.)

Question 3: How important is CAS

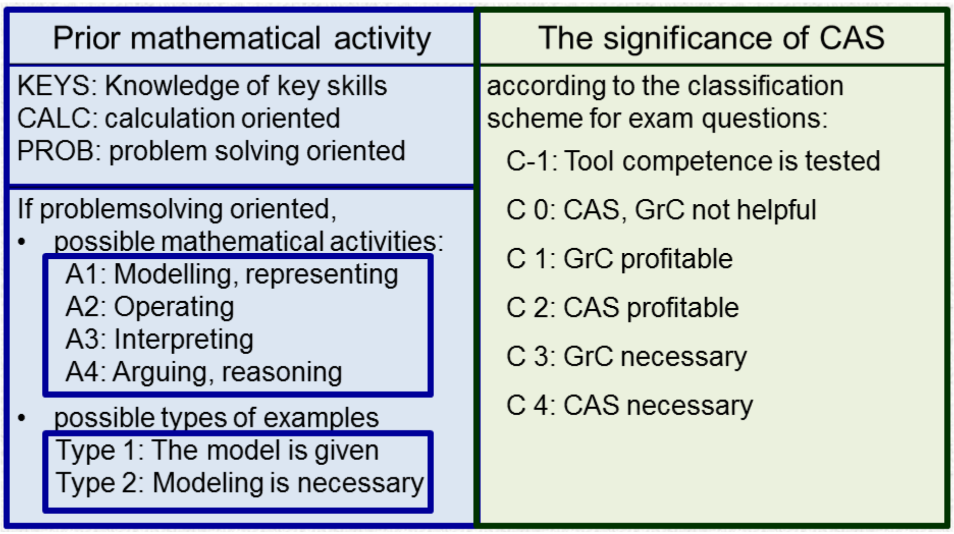
To characterize the role of CAS I have used a classification scheme for exam questions which was at first published in the book “The Case for CAS” [Böhm, a.o., 2004]. While in this book 5 catogories are used, my classification scheme consists of 6 categories. I have added category C-1



2. Investigation of examples of final exams and conclusions for the mathematics education

I have investigated two groups of aspects:

- Prior mathematical activities which are significant for the examples
- The role of CAS for modeling, the problem solving process and for documenting and arguing the results.



Results of 5 countries:

Denmark

In Denmark CAS is obligatory since many years. Students use computers, tablets or handhelds.

The final exam consists of 2 parts:

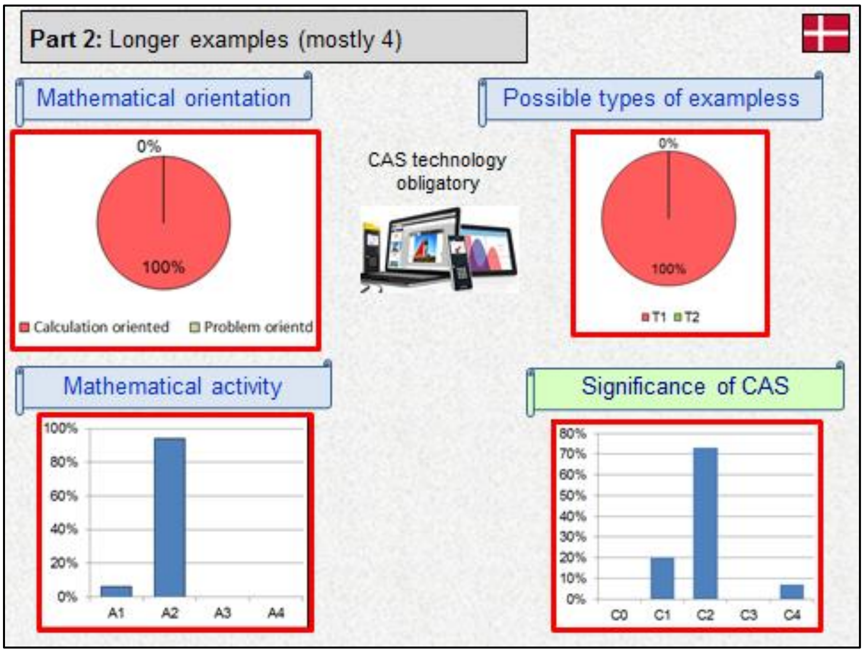
Part A: Without any technology. About 7 short examples are given examining key skills. especially the examples which examine calculation competence are examples of category C-1 :

Example 1 (MAT 3 Part A): Solve the equation ln(3.x – 14) = 0

The solution of a CAS tool: $\text{solve}(\ln(3\cdot x-14)=0,x) \qquad x=5$

Tested is only the competence to handle the tool.

Part B: CAS is obligatory. About 4 longer examples are given.



It is surprising that most of the examples are calculation oriented. Almost all examples are of Type T1 that is the mathematical model is given. Calculating is the dominating activity.

Many examples are of category C2, they can be solved better and faster with CAS. Few examples can only be solved by using CAS (C4)

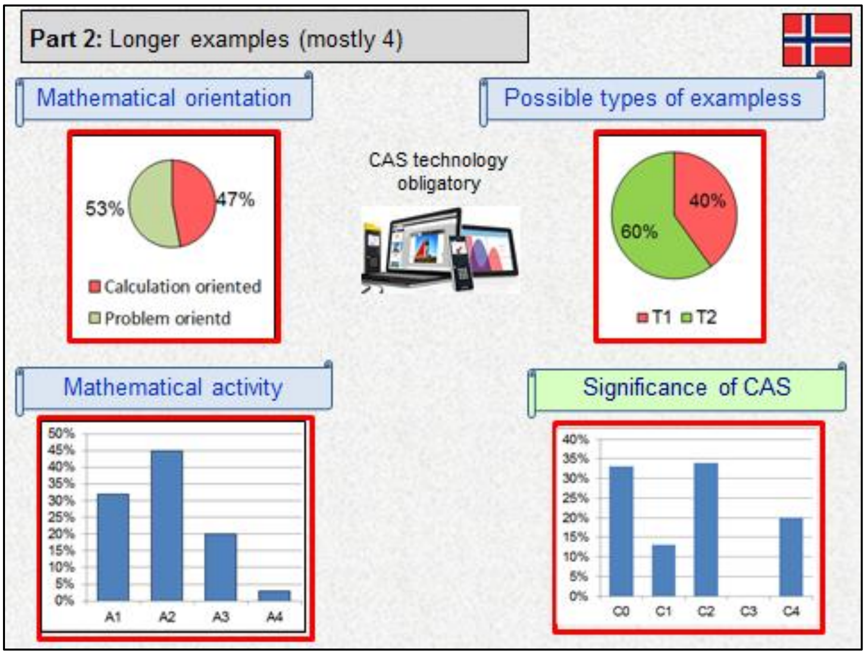
Norway

In Norway CAS is obligatory.

The final exam consists of 2 parts:

Part A: Without any technology. About 7 short examples are given examining key skills. Especially the examples which examine calculation competence are examples of category C-1 :

Part B: CAS is obligatory. About 4 longer examples are given.



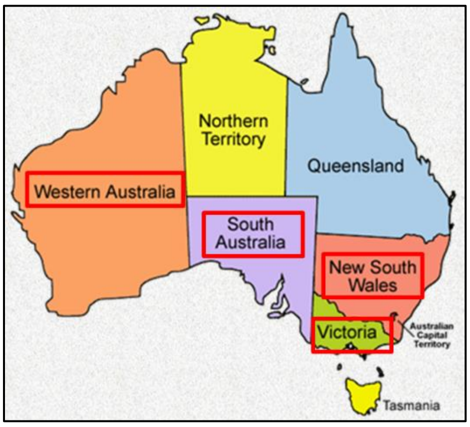
More than 50% of the examples are problem solving oriented.

In the majority of the examples students have to find the mathematical model (Type T2).

All sorts of mathematical activities can be found.

The category C0 is typical for stochastic examples. Also more examples of category C4 can be found which only can be solved by using CAS.

Australia



Examples of 4 Australian states have been analyzed. Following sorts of technology are used:

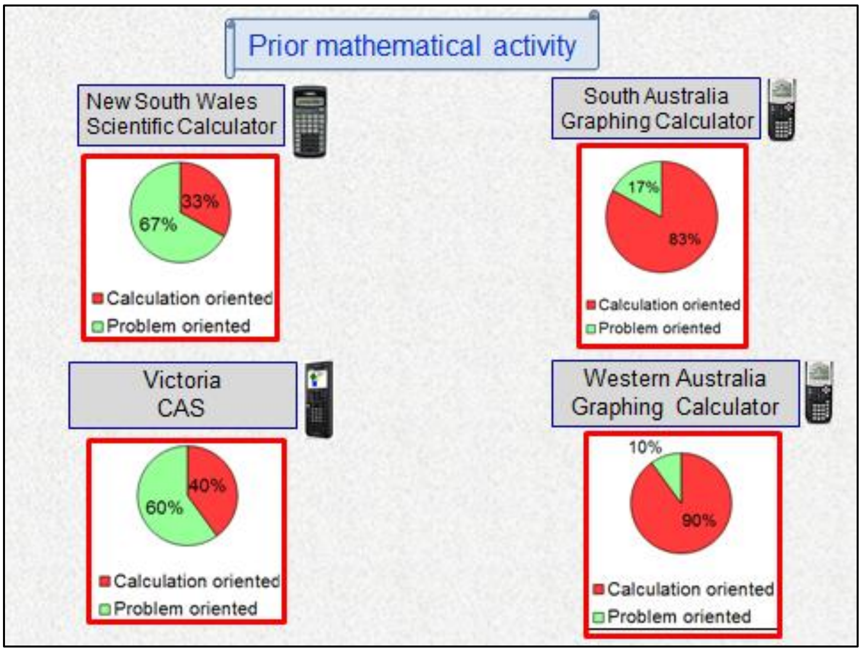
- New South Wales ⇔ Scientific Calculators
- South Australia ⇔ Graphing Calculators
- Western Australia ⇔ Graphing Calculators
- Victoria ⇔ CAS

From several mathematics courses I have focused on “Mathematics Methods” with the emphasis Algebra and Analysis.

Most of the external exams have two parts:

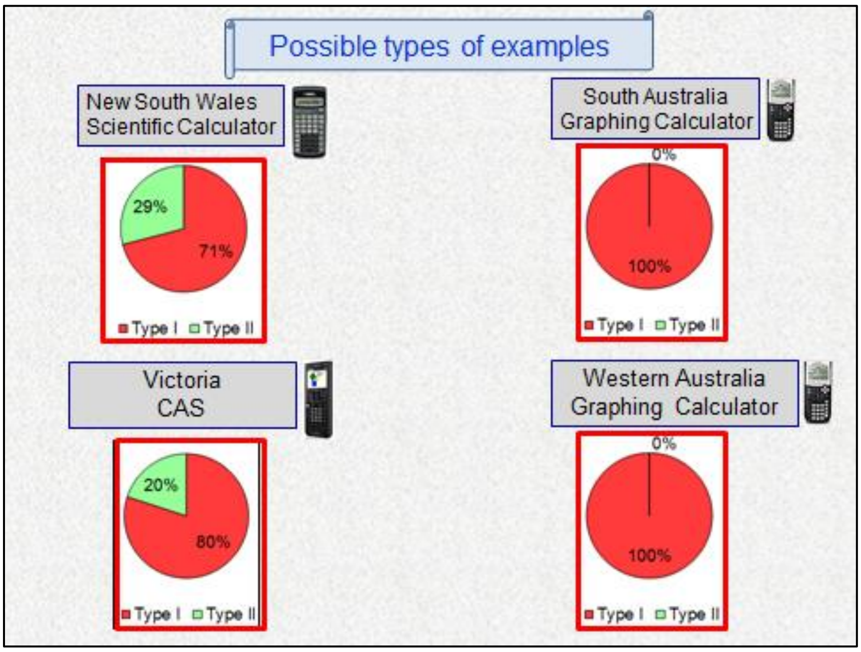
Part 1: Shorter examples (sometimes in multiple choice format) examining key skills. No technology is used.

Part 2: Longer examples with necessary use of technology.



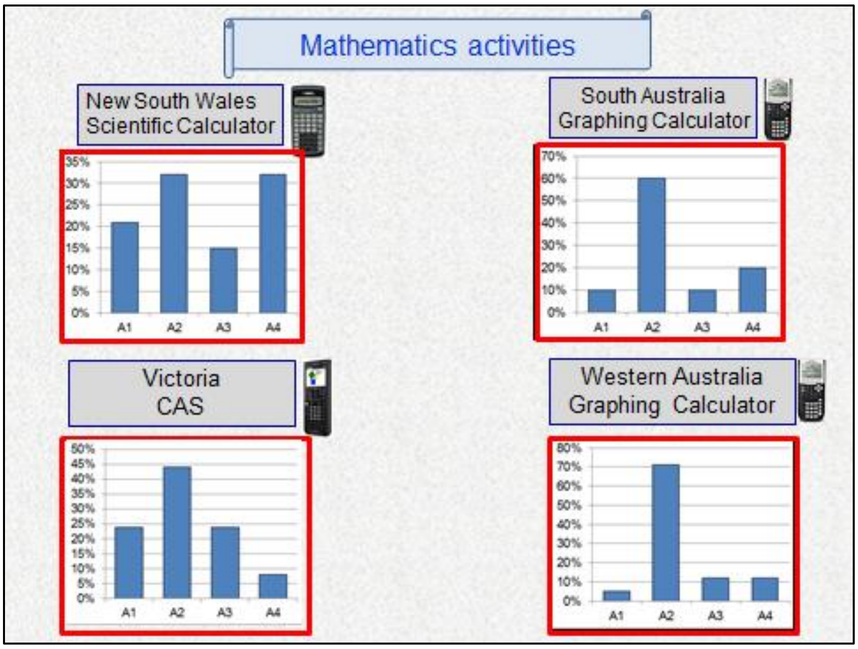
Interesting is that the state where only scientific calculators are allowed has the highest percentage of problem solving oriented examples while in the two “graphing calculator-states” calculation orientation is dominating.

Also in Victoria where CAS is obligatory more examples are problem solving oriented.

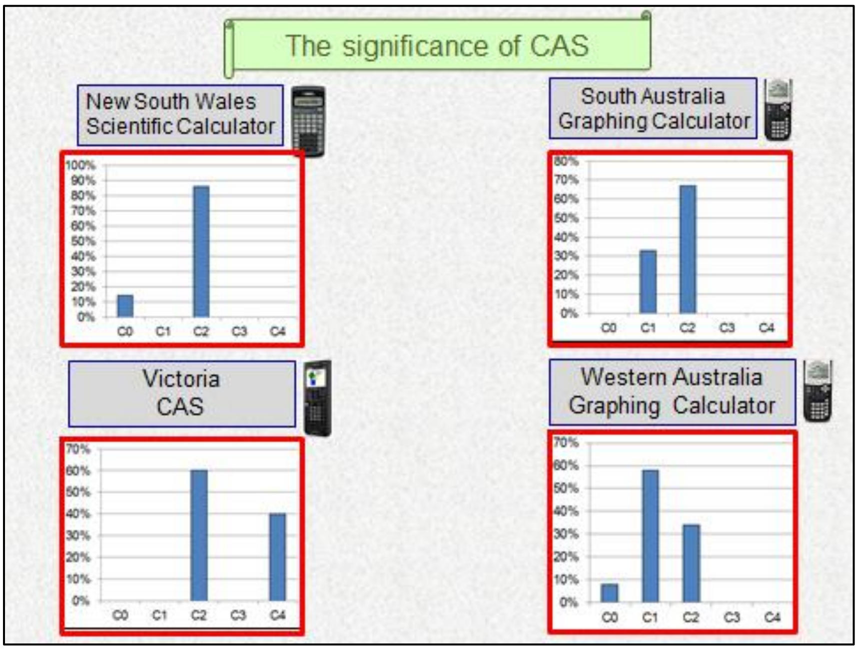


In all states examples of type T1 are dominating. The mathematical models are given.

In New South Wales (scientific calculators) and in Victoria (CAS) one can also find Type T2, examples where students have to develop the mathematical model.



In New South Wales (scientific calculators) and in Victoria (CAS) one can find a good distribution of all mathematical activities while in South Australia and Western Australia (graphing calculators) operating is dominating.



It is not surprising that in New South Wales examples which are developed for scientific calculators can be solved faster or even trivialized by using CAS.

Also the distribution in states with graphing calculators is to be expected – partly with advantages of graphing calculators and partly with better solutions by using CAS.

In Victoria (CAS) also examples can be found which only can be solved with CAS (C4).

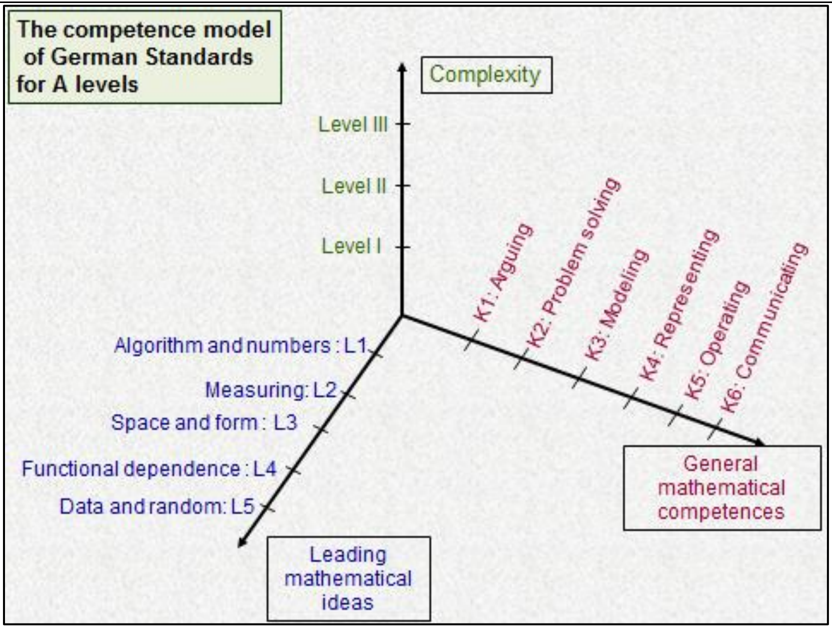
Germany

All of the 16 German states have their individual final exam. At present there are first attempts to develop examples for a common final exam.

The conference of the educational ministers (KMK) and the institute for quality development of the educational system (IQB) developed standards for final exams and samples of exam examples.

Authorized technology:

- 1 State: Until 2019 graphing calculators, than only scientific calculators
- 2 States only CAS
- 13 States either scientific calculators and CAS or graphing calculators and CAS with special examples for each type of technology.



The German competence model is similar to the models of NCTM and of Austria.

The contents are specified as “leading mathematical ideas” and the “general mathematical competences” are characterized by 6 mathematical activities.

Characteristic of German exams: Mostly 2 parts:

- Part 1: Shorter examples examining key skills (no technology allowed)
- Part 2: Longer examples; problem oriented. As many aspects of mathematical competences as possible should be considered (K1, K2, K3, K4, K5, K6)

Characterization of expected mathematical competences for an example of the IQB-collection for all German states:

GERMANY IQB Analysis Ex. 1 fundamental level

Mathematical ideas (contents) | Mathematical competences (activities)

Teil-aufg.	BE	Leitideen					allgemeine mathematische Kompetenzen[1]						Anforderungs-bereich		
		L1	L2	L3	L4	L5	K1	K2	K3	K4	K5	K6	I	II	III
a	4	X	X	X	X		I		I		I		X		
b	4		X	X	X				I	I	I		X		
c	3	X	X	X	X				I		I		X		
d	5	X	X	X	X			III	III			II			X
e	6	X	X	X	X			II	II		II			X	
f	2			X	X					II		I		X	
g	4	X		X	X			III	II		II				X
h	3			X	X		II		I			II		X	
i	2				X					I			X		
j	3	X		X	X		II	II			II			X	
k	4	X	X	X	X			II	II		II			X	

All of the 6 characteristics of mathematical competences should appear in this example.

Austria

Since 2015 we have a „standardized competence oriented zentral final exam“

The only goal is to examine „key skills“ or in our diction „fundamental competences“.

Influenced by psychometricians the exam consists of two parts:

Part 1: 24 short „0/1 items“ mostly in multiple choice or other closed formats

Part 2: Exists of 4 theoretically longer examples. In reality they consist of about 4 short independent subitems similar to the examples of part 1.

Admitted technology: Every sort of technology which is used in the classroom is also allowed in the final exam (Scientific calculators as well as CAS)

Classification according the role of technology:

Most of the examples are "C0-examples", where neither graphing calculators nor CAS is helpful. Bur one can find also examples of type C-1 where students of technology classes have advantages. However tested is in technology classes tool competence and not the intended mathematical competence.

An example of type C-1:

AUSTRIA BIFIE
Part 1: Ex. 11

A real function f with $f(x) = c \cdot a^x$ with $c \neq 0$ and $a > 0$ Is an exponential function.

Mark the two cases which are true for this exponential function for all values $k, h \in \mathbb{R}$ and $k > 1$

$f(k \cdot x) = k \cdot f(x)$	☐
$\frac{f(x+h)}{f(x)} = a^h$	☐
$f(x+1) = a \cdot f(x)$	☐
$f(0) = 0$	☐
$f(x+h) = f(x) + f(h)$	☐

$f(x) := c \cdot a^x$	*Done*
$f(k \cdot x) = k \cdot f(x)$	$c \cdot a^{k \cdot x} = c \cdot k \cdot a^x$
⚠ $\frac{f(x+h)}{f(x)} = a^h$	true
$f(x+1) = a \cdot f(x)$	true
⚠ $f(0) = 0$	$c = 0$
$f(x+h) = f(x) + f(h)$	$c \cdot a^{h+x} = c \cdot a^x + a^h \cdot c$

The 5 conditions are expressed in the mathematical language and can therefore directly entered into the CAS tool.

And then the CAS tool decides if a condition is true or not and not the student..

An example of the second part of the exam:

Example Austria 2016 Part 2 Ex.3: Income tax

Employed persons must pay a certain amount of their income to the state. The tax model of 2015 differentiates between 4 tax classes with the tax rates 0%, 36,5%, 43,2%, and 50%.

Model assumption: „*Yearly net income = taxable yearly income – income tax*“

In July 2015 a new tax law was enacted: This starting with January 2016 valid tax model is a model with 7 tax classes (0%, 25%, 35%, 43%, 50%, 55%). The graphic shows the model with 4 tax classes (valid until 2015) and the model with 7 tax classes (valid from 2016).

The graph with the appropriate tax classes can be seen on the next page (figure 1.)

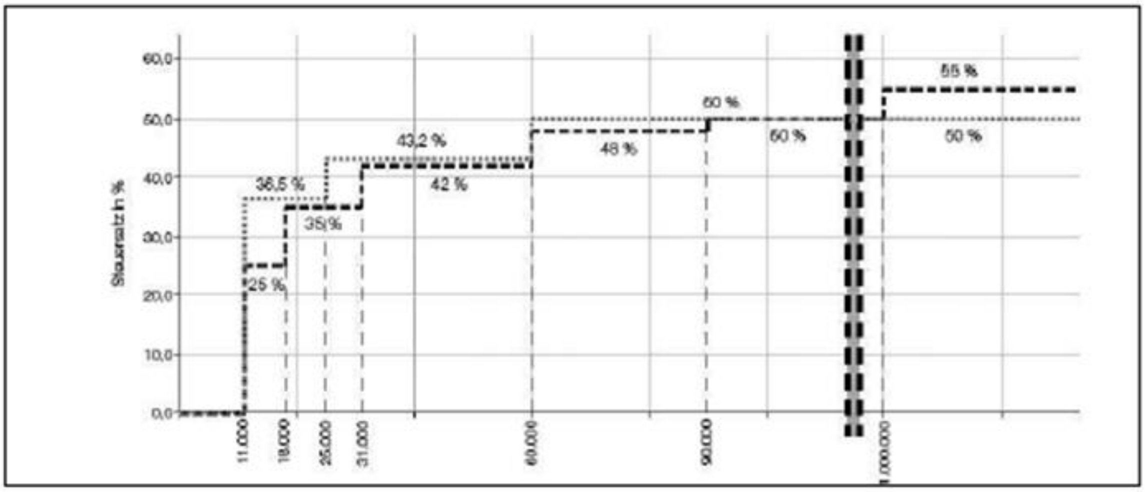


figure 1.

Comment: A very interesting context and a good contribution of mathematics to higher general education. But let us look at the expected mathematical competences:

Expected answers/competences:

Teil (a)

20 000 – 9 000 · 0,365 = 16 715 ⇒ € 16.715 — 1 A

N = *E* – (*E* – 11 000) · 0,365 — 1

Calculating with decimal numbers

Teil (b)

$$\frac{14000 * 0.365 + 15000 * 0.432}{40000} \approx 0.29 \Rightarrow 29\%$$ — 1

Percentage calculation

Mit dem Term wird die Steuerersparnis (in Euro) dieser Person durch das neue Steuermodell (im Vergleich zum 2015 gültigen Modell) berechnet. — 1

Reading competence

Teil (c):

Beide Behauptungen sind falsch.
Die Einkommensanteile unter € 90.000 sind geringer. — 1

Änderung um 11,5 *Prozentpunkte*, das sind $\frac{11,5}{36,5} \approx 31,5$ *Prozent*. — 1

Teil (d)

$$\frac{15125}{35000} \approx 0.432$$ — 1

5 110 ist die Einkommensteuer für die ersten € 25.000 — 1

Only competences of secondary level I are tested (percentage calculation, calculation with decimal numbers and above all “reading competence”).

The right column shows that it is not a problem oriented task it consists of 8 independent 0/1 items testing fundamental competence.

A vision of a problem oriented example for CAS classes with this context:

CAS Example Austria: Income tax

The context is the same as in the example before: Two tax models (“2015” and “2016”) with different tax classes (see figure 1.)

The task:

a) Develop 2 mathematical models for the income tax namely for the tax model valid until 2015 and the new tax model valid up 2016 with respect to the taxable yearly income. Draw the graphs in the interval [0€, 200000€]. Look for a formula for the yearly net income for both tax models.

b) The political order was, to discharge lower earnings fiscally and to charge higher earnings. Discuss by calculating and graphically if the new tax law is coming up to the political expectations. From what taxable yearly income the tax burden is higher with the new model?

A solution by using TI Nspire:

Model 1 valid until 2015 with 4 tax classes

$$est15(x):=\begin{cases}0, & 0\le x\le 11000\\ 0+(x-11000)\cdot 0.365, & 11000<x\le 25000\\ 0+14000\cdot 0.365+(x-25000)\cdot 0.432, & 25000<x\le 60000\\ 0+14000\cdot 0.365+35000\cdot 0.432+(x-60000)\cdot 0.5, & x>60000\end{cases}$$

Done

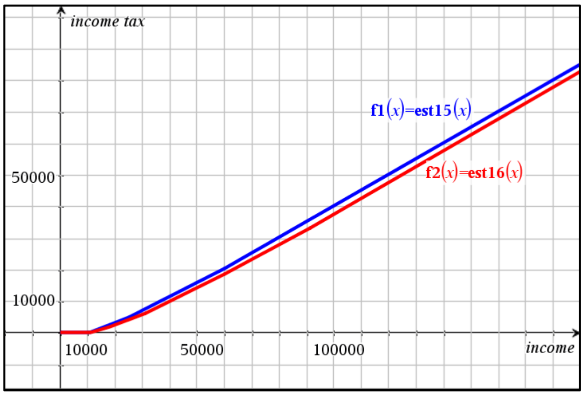
Model 2 valid from 2016 with 7 tax classes

$$est16(x):=\begin{cases}0, & 0\le x\le 1100\\ 0+(x-11000)\cdot 0.25, & 11000<x\le 18000\\ 0+7000\cdot 0.25+(x-18000)\cdot 0.35, & 18000<x\le 31000\\ 0+7000\cdot 0.25+13000\cdot 0.35+(x-31000)\cdot 0.42, & 31000<x\le 60000\\ 0+7000\cdot 0.25+13000\cdot 0.35+29000\cdot 0.42+(x-60000)\cdot 0.48, & 60000<x\le 90000\\ 0+7000\cdot 0.25+13000\cdot 0.35+29000\cdot 0.42+30000\cdot 0.48+(x-90000)\cdot 0.5, & 90000<x\le 1000000\\ 0+7000\cdot 0.25+13000\cdot 0.35+29000\cdot 0.42+30000\cdot 0.48+910000\cdot 0.5+(x-1000000)\cdot 0.55, & x>1000000\end{cases}$$

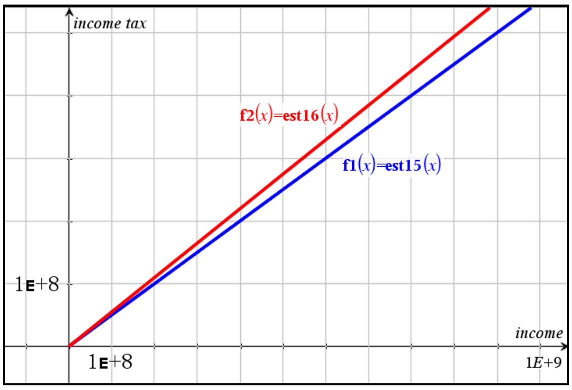
Function values of the two models for any earnings can be calculated

$est15(100000)$	40230.
$est16(100000)$	37880.

The graph is a good possibility to answer the questions of part (b)



Interval [0 €; 200.000 € [

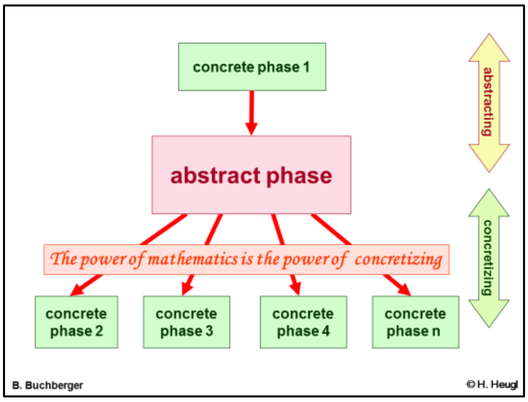


Interval [100.000.000 €; 1.000.000.000 € [

From what taxable yearly income the tax burden is higher with the new model?

⚠ $\text{solve}(est16(x)=est15(x),x)$	$x=0.$
$\text{solve}(0+7000\cdot 0.25+13000\cdot 0.35+29000\cdot 0.42+30000\cdot 0.48+910000\cdot 0.5+($ ▸	$x=1.047\text{E}6$
⚠ $\text{solve}(est15(x)=est16(x)\vert x>100000,x)$	$x=1.047\text{E}6$

Comment: This example is especially interesting because it shows the learners the typical mathematical way when solving a problem or developing a new mathematical field.



This way consists of 2 phases:

- *the phase of abstracting and*
- *the phase of concretizing*

After starting with a concrete problem in the abstract phase an abstract model is developed. This abstract model opens the door for solving many concrete problems.

The two abstract tax models can used for answering all the questions of the example "income tax"

Conclusions of the investigation

1. CAS supports the shift from a calculation oriented to a problem solving oriented mathematics education. Reasons:
   - ☞ a shift from doing to planning
   - ☞ the direct translation of verbal information into the language of mathematics
   - ☞ the execution of complex operations
   - ☞ the reduction of complexity by using names of complex expressions instead of the complex expressions

   Examples:

Australia Victoria Part 2 Example 2 Analysis See page 16	GERMANY IQB Analysis Ex. 1 fundamental level See page 19	GERMANY Thüringen Part B Ex.2 Analysis See page 22

2. Problem solving examples can also be found in countries where only scientific or graphing calculators are allowed, but
   - ☞ from all problem solving activities (modeling, calculating, interpreting, arguing) calculating is dominating,
   - ☞ when using CAS for such examples they could be solved faster or even trivialized (type C2)

AUSTRALIA NSW Section 2: EX.12 See page 24	AUSTRALIA SA Part 2 Ex. 16 See page 28

3. A dominant calculation orientation can also be found in countries where CAS is obligatory, but
   - ☞ the consequence of which is that more tool competence than mathematical competence is checked and
   - ☞ mathematical competences like modeling, arguing and reasoning are neglected

Denmark Europ. Exam Part 2 Ex. 1 See page 30

4. In most of the problem solving examples the model of the applied problem is given („example of type T1"). Modeling i. e. looking for a mathematical model based on data or verbal information is rather unusual („example of type T2"). Modeling is more often necessary for finding certain solutions, e.g. extrema, inflection points.
But technoloy supports modeling:
    - CAS allows the construction of new mathematical language elements which can be used to develop mathematical models.
    - When using these language elements CAS allows a direct translation of verbal information into the language of mathematics.
    - Using regression functions mathematical models suitable to functional relations of data can be found.
    - Difference or differential equations can be used to analyze dynamic systems.

NORWAYAY Ex 4 Part 2 T2 See page 32	NORWAYAY Ex 3 Part 2 T2 See page 33	Australia Victoria Part 2 Ex. 2 T1 => T2 See page 34

5. In many countries in the first part of final exams „key skills" or „fundamental competence" are tested in short examples. Mostly no technology is allowed.
    - In this case it is better not to allow technology and especially CAS because the tool impedes the examination of the intended mathematical competence. Instead of that skills for handling the tool will be tested.

Denmark Europ. Exam Part 1 Ex. 1	1) Berechnen Sie $\int_1^e 2\cdot\frac{\ln(x)}{x}dx$.	$\int_1^e \left(2\cdot\frac{\ln(x)}{x}\right)dx$ 1 **C-1**

6. Few examples are of type C4 can be found (examples which only can be solved by the use of CAS). Type C4 often means that the task or the needed calculations are too complex and therefore CAS is essential. Most of the examples are of type C2, these are traditional examples which are solved faster or even trivialized by CAS.
This result is not surprising, because the use of technology does not change the mathematical contents (except recursive models). Therefore most of the given problems could also be solved without CAS. The advantages of CAS are the characteristic ways of thinking and calculating. CAS does not only support cognition it becomes part of cognition [Dörfler 1991].

NORWAYAY Ex 4 Part 2 C4 See page 32	NORWAYAY Ex 3 Part 2 C4 See page 34	GERMANY IQB Analysis Ex. 1 fundamental level See page 19

7. In application oriented examples the understanding of the context is sometimes more difficult than the mathematical problem:

- ☞ The authors of such examples must consider that the learners often are not familiar with such an application field. Sometimes additional information concerning the context is necessary.

GERMANY Thüringen Part B Ex.2 See page 36	AUSTRIA 2016 Part 2: Ex 3 See page 10

8. A mistake of the authors of exams in CAS-classes is that the examples mostly are more difficult and more complex as the examples of classes where traditional tools are used. Sometimes additional mathematical contents were tested in CAS classes. The authors often try to show the full power of CAS which also makes the tasks more difficult.
   Following aspects should be considered when using CAS:
   - ☞ When using CAS we should show the learners that fascinating problems can be solved easier and faster than without CAS.
   - ☞ The higher complexity of expressions caused by the use of CAS in such examples are not a better contribution to the goals of mathematics education. Students do not work with these expressions, they use the names of the expressions. Therefore their cognitive activities are the same as in tasks with less complex terms.
   - ☞ More complex expressions are only senseful to describe a more complex reality in applied problems.

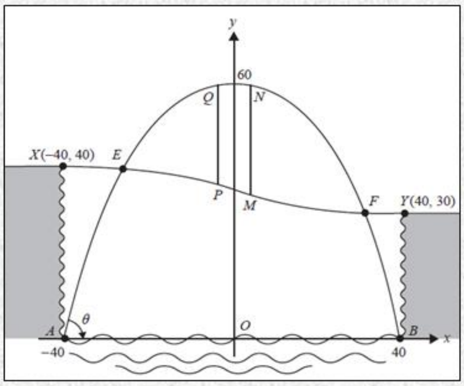
Summary:

- ➧ The partition of most of the final exams in two sections, one where no technology is allowed and the second one with technology is useful. In the first section fundamental competence including calculation competence can be tested. If technology would be allowed tool competence will be tested instead of the intended mathematical competence. The use of technology and especially the use of CAS in part two causes a shift to more problem solving orientation.
- ➧ In a calculation oriented math education and therefore in calculation oriented tests CAS is not necessary - students should calculate „by hand“
- ➧ In a problem solving oriented math education CAS is indispensable. The execution of complex operations by the tool and the availability of new and better mathematical models support the problem solving orientation.
- ➧ The better we will try to describe reality the more complex the mathematical models will be. Such complex models and the following complex calculations need the use of CAS for solving those problems.

Examples

Australia Victoria
Part 2 Example 2
CAS

PROB | C 4 | A 1 | A 2 | A 3 | A 4



A city is located on a river that runs through a gorge. The gorge is 80 m across, 40 m high on one side and 30 m high on the other side.
A bridge is to be built that crosses the river and the gorge. A diagram for the design of the bridge is shown below.

The main frame of the bridge has the shape of a parabola.

The parabolic frame is modelled by $y = 60 - \frac{3}{80} \cdot x^2$

and is connected to concrete pads at *B* (40, 0) and *A*(– 40, 0). The road across the gorge is modelled by a cubic polynomial function.

a) Find the angle, *θ*, between the tangent to the parabolic frame and the horizontal at the point *A*(– 40, 0) to the nearest degree.

The road from *X* to *Y* across the gorge has gradient zero at *X* (– 40, 40) and at *Y* (40, 30), and has equation $y = \frac{x^3}{25600} - \frac{3x}{16} + 35$

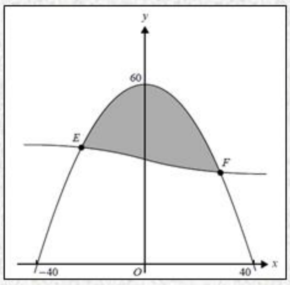
b) Find the maximum downwards slope of the road. Give your answer in the form $-\frac{m}{n}$ where *m* and *n* are positive integers.

Two vertical supporting columns, *MN* and *PQ*, connect the road with the parabolic frame.
The supporting column, *MN*, is at the point where the vertical distance between the road and the parabolic frame is a maximum.

c) Find the coordinates (*u*, *v*) of the point *M*, stating your answers correct to two decimal places.

The second supporting column, *PQ*, has its lowest point at *P* (–*u*, *w*).

d) Find, correct to two decimal places, the value of *w* and the lengths of the supporting columns *MN* and *PQ*.

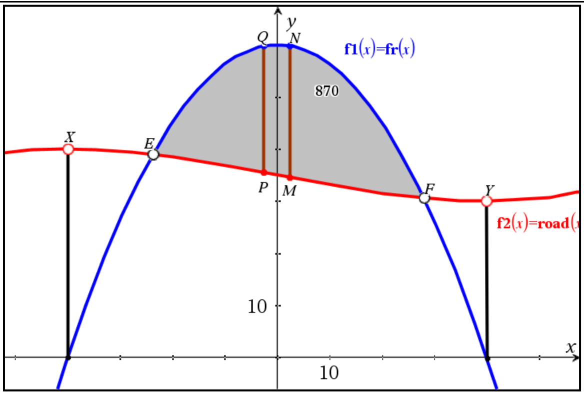


For the opening of the bridge, a banner is erected on the bridge, as shown by the shaded region in the diagram below.

e) Find the *x*-coordinates, correct to two decimal places, of *E* and *F*, the points at which the road meets the parabolic frame of the bridge.

f) Find the area of the banner (shaded region), giving your answer to the nearest square meter.

$fr(x):=60-\frac{3}{80}\cdot x^2$ — Done $fr1(x):=\frac{d}{dx}(fr(x))$ — Done $\tan^{-1}(fr1(^-40.))$ — 71.5651 a) The angle between the parabolic frame and the horizontal: $\theta = 72°$	$road(x):=\frac{x^3}{25600}-\frac{3\cdot x}{16}+35$ — Done $road1(x):=\frac{d}{dx}(road(x))$ — Done $road2(x):=\frac{d}{dx}(road1(x))$ — Done $\text{zeros}(road2(x),x)$ — $\{0\}$ $road1(0.)$ — $^-0.1875$ $road1(0)$ — $\frac{-3}{16}$ b) The maximum downward slope of the road: $k_r = -19\%$ or $k_r = -\frac{3}{16}$
$dist(x):=fr(x)-road(x)$ — Done $dist1(x):=\frac{d}{dx}(dist(x)$▸ — Done $\text{zeros}(dist1(x),x)$ — $\{^-40\cdot(\sqrt{65}+8),40\cdot(\sqrt{65}-8)\}$ $\{^-40.\cdot(\sqrt{65}+8),40.\cdot(\sqrt{65}-8)\}$ — $\{^-642.49,2.49031\}$ c) The maximum vertical distance between the road and the frame is at the point M with x = 2.49031	$m:=\begin{bmatrix}2.49\\34.53\end{bmatrix}$ — $\begin{bmatrix}2.49\\34.53\end{bmatrix}$ $n:=\begin{bmatrix}2.49\\fr(2.49)\end{bmatrix}$ — $\begin{bmatrix}2.49\\59.7675\end{bmatrix}$ $p:=\begin{bmatrix}^-2.49\\35.47\end{bmatrix}$ — $\begin{bmatrix}^-2.49\\35.47\end{bmatrix}$ $q:=\begin{bmatrix}^-2.49\\fr(^-2.49)\end{bmatrix}$ — $\begin{bmatrix}^-2.49\\59.7675\end{bmatrix}$ $dist(2.49031)$ — 25.2338 $dist(^-2.49031)$ — 24.3011 The coordinates of the final points of vertical supporting columns MN and PQ. d) The length of the supporting columns: MN = 25.23 m PQ = 24.30 m
$\text{solve}(fr(x)=road(x),x)$ $x=^-964.289$ or $x=^-23.7068$ or $x=27.9963$ $e:=\begin{bmatrix}^-23.7068\\fr(^-23.7068)\end{bmatrix}$ — $\begin{bmatrix}^-23.7068\\38.9245\end{bmatrix}$ $f:=\begin{bmatrix}27.9963\\fr(27.9963)\end{bmatrix}$ — $\begin{bmatrix}27.9963\\30.6078\end{bmatrix}$ $area:=\int_{^-23.7068}^{27.9963}(fr(x)-road(x))dx$ — 869.619	e) The intersection points between the road and the parabolic frame: E(-23.71, 38.92) F(28.00, 30.61) f) The area of the banner: A = 570 m^2

	The graph of the given problem

IQB sample of examples for a common final exam in CAS-classes

GERMANY IQB
Analysis Ex. 1
fundamental level
CAS

PROB | A 1 | A 2 | A 3 | A 4 | C 4

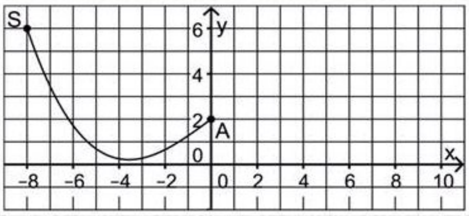


Figure 1

A ramp for BMX bikes will be installed. The limit line can be defined by the polynomial function f with

$$f(x) = -\frac{6}{256}x^3 + \frac{3}{4}x + 2 \text{ with } x \in [-8, 0]$$

The startig point S(-8, f(-8)) and the jump-off point A(0, f(0)) are given. The surface level is identical with the x-axis in the diagram..Scale unit is 1 m.

a) Look for the coordinates of the lowest point of the ramp and calculate the vertical height between the highest and the lowest point.

b) Draw the graph of the ramp and visualize the average slope between S and A.

c) Determine the angle at the starting point S between the ramp and the horizontal line.

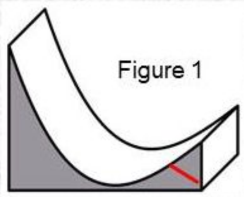


Figure 2 shows the grey side surface of the ramp.

d) Because of the stabilazation of the ramp a steel stay is necessary from the down right point to the limit line f. Calculate the minimum length of the steel stay and draw the steel stay in the graph.

e) The part of the side surface which is more than 2 m above the ground is used for advertising. Calculate the proportion of this area and the whole side surface.

Adjacent to the ramp a hill suitable for the landing is built. A possible model of the limit line of the hill is a function ga with $ga(x) = x \cdot e^{-a \cdot x^2}$ with $x \geq 0$ and $a \in \mathbb{R}^+$

f) Look for the value of a so that the highest point of the hill has the same height as the jump-off point A.

g) Take for a = $\frac{1}{24}$ and draw the suitable graph ga. Determine the point of ga with the largest downward slope.

h) A possible model of the trajectory of the BMX biker is the quadratic fubction hb with $hb(x) = b \cdot x^2 + \frac{3}{4} \cdot x + 2$. The biker is landing on the hill with horizontal distance of 6 m. Determine b. Calculate the largest height of the trajectory relating to the jump-off point.

a)

The coordinates of the lowest point of the ramp exact and also the approximate solution

$f(x):=\frac{-5}{256}\cdot x^3+\frac{3}{4}\cdot x+2$	*Done*
$f(-8)$	6
$f1(x):=\frac{d}{dx}(f(x))$	*Done*
$\text{zeros}(f1(x),x)$	$\left\{\frac{-8\cdot\sqrt{5}}{5},\frac{8\cdot\sqrt{5}}{5}\right\}$
$f\left(\frac{-8\cdot\sqrt{5}}{5}\right)$	$2-\frac{4\cdot\sqrt{5}}{5}$
$\left\{\frac{-8.\cdot\sqrt{5}}{5},\frac{8.\cdot\sqrt{5}}{5}\right\}$	$\{-3.57771,3.57771\}$
$f\left(\frac{-8.\cdot\sqrt{5}}{5}\right)$	0.211146

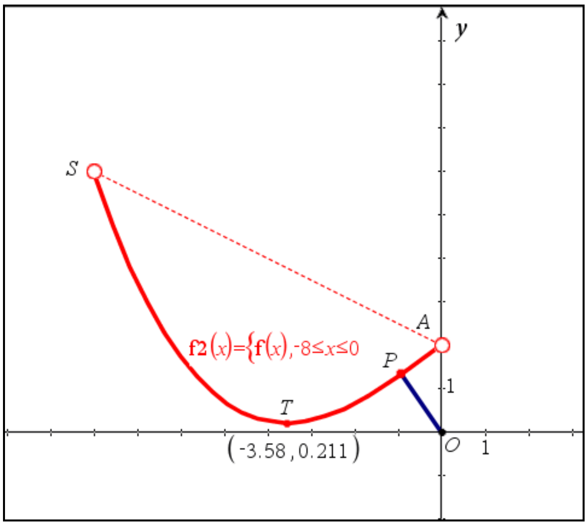
The height between the highest and the lowest point is 5.8m. The angle at the starting point S is -71.57°.

$f(-8.)-f\left(\frac{-8\cdot\sqrt{5}}{5}\right)$	5.78885
$\tan^{-1}(f1(-8.))$	-71.5651

The coordinates of the endpoint of the steel stay P(-0.93, 1.32).

$dist(x):=\sqrt{x^2+(f(x))^2}$	*Done*
$dist1(x):=\frac{d}{dx}(dist(x))$	*Done*
$\text{zeros}(dist1(x),x)$	$\{-0.925011\}$
$f(-0.925011)$	1.3217

The graph of the ramp inclusive the steel stay and the lowest point



e)

The area which is used for advertising

The whole area of the side surface

The proportion

The percentage: 27% of the whole area is used for advertising

$\text{solve}(f(x)=2,x)$	$x=\frac{-8\cdot\sqrt{15}}{5}$ or $x=0$ or $x=\frac{8\cdot\sqrt{15}}{5}$
$aad:=\int_{-8}^{\frac{-8\cdot\sqrt{15}}{5}}(f(x)-2)dx$	$\frac{16}{5}$
$aju:=\int_{-8}^{0}f(x)\,dx$	12
$\frac{aad}{aju}$	$\frac{4}{15}$
$\frac{4.}{15}$	0.266667

f)

The value of a so that the highest point of the hill has the same height as the jump-off point A:

$$a=\frac{e^{-1}}{8}$$

$ga(x):=x\cdot e^{-a\cdot x^2}$	Done
$ga1(x):=\frac{d}{dx}(ga(x))$	Done
$\mathrm{zeros}(ga1(x),x)$	$\left\{\left\{\frac{\sqrt{2}}{2\cdot\sqrt{a}},\frac{1}{a}\geq 0\right\},\left\{\frac{-\sqrt{2}}{2\cdot\sqrt{a}},\frac{1}{a}\geq 0\right\}\right\}$
⚠ $ga\left(\frac{\sqrt{2}}{2\cdot\sqrt{a}}\right)$	$\frac{e^{\frac{-1}{2}}\cdot\sqrt{2}}{2\cdot\sqrt{a}}$
$xgam:=\frac{1}{\sqrt{2\cdot a}}$	$\frac{\sqrt{2}}{2\cdot\sqrt{a}}$
⚠ $\mathrm{solve}(ga(xgam)=2,a)$	$a=\frac{e^{-1}}{8}$

g)

The point of the hill with the largest down-wardslope is the inflection point W(6, 1.37).

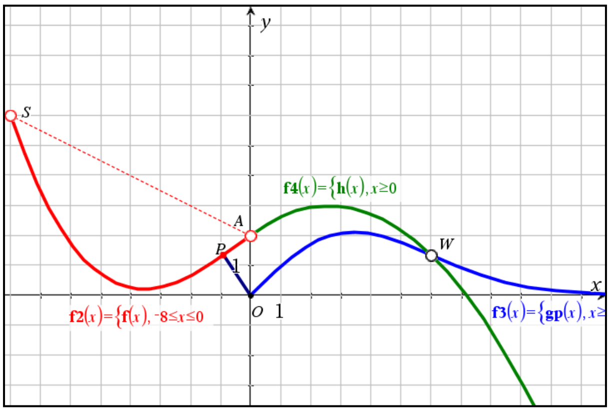
$gp(x):=ga(x)\vert a=\frac{1}{24}$	Done
$gp1(x):=\frac{d}{dx}(gp(x))$	Done
$gp2(x):=\frac{d}{dx}(gp1(x))$	Done
$\mathrm{zeros}(gp2(x),x)$	$\{-6,0,6\}$
$gp(6.)$	1.33878

h)

The model of the trajectory of the BMX biker

$hb(x):=b\cdot x^2+c\cdot x+d$	Done
$hb1(x):=\frac{d}{dx}(hb(x))$	Done
$hb(0)=2$	$d=2$
$hb1(0)=f1(0)$	$c=\frac{3}{4}$
$hb(x):=b\cdot x^2+0.75\cdot x+2$	Done
$\mathrm{solve}(hb(6.)=gp(6),b)$	$b=-0.143367$
$h(x):=hb(x)\vert b=-0.143367$	Done

The visualization of the problem



Thüringen Final exam 2016 CAS obligatory

PROB | A 1 | A 2 | A 3 | A 4 | C 4

GE Thüringen
Part B Ex.2
Analysis
CAS

Figure 1

A specific type of a roof is called „dormer“ (see figure 1).

The proportion of height and width should be between 1:5 and 1:6.

A possible model of the border line is: $f(x) = \frac{4}{3 \cdot x^2 + 4} - \frac{1}{4} \quad (x_1 \le x \le x_2)$

X_1 and x_2 are the zeros, length in m.

a) Show that the graph of f is symmetrical relating to the y-axes.

b) Investigate if the proportion of height and width is complying with the standard.

c) At both ends of the dormer the downward slope should not be larger than 12°. Complies the model f this condition? Calculate the points whith the largest downward slopes.

A parabolic window with a height h = 0.5 m and a width b should be installed. The lower frame is horizontal and the upper frame is a parabola with the equation $p(x) = c \cdot x^2 + 0.5 \quad (c \in \mathbb{R})$

d) The width of the window should be 2 m. Calculate the area of the window.

f) A function fa is given for any positive real number a with $fa(x) = \frac{4}{a \cdot x^2 + 4} - \frac{1}{4} \quad (x \in \mathbb{R})$

Determine the parameter a so that the proportion of height and width of the dormer will be between 1:5 and 1:6.

g) In Asian countries such dormers often look like a pagoda. In Figure 2 one can see the border line of the roof of such a pagoda.

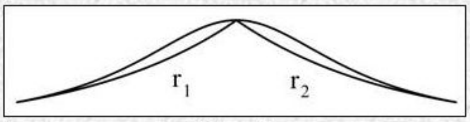


Figure 2

Find the two function equations of the graphs r_1 and r_2

a)

Zeros: x1 = -2, x2 = 2.

f is symmetrical relating to the y-axes

b)

The proportion of height and width is complying with the standard.

$f(x):=\frac{4}{3\cdot x^2+4}-\frac{1}{4}$	*Done*
$\text{zeros}(f(x),x)$	$\{-2,2\}$
$f(-x)=f(x)$	true
$f(0)$	$\frac{3}{4}$
$\frac{f(0)}{4}$	$\frac{3}{16}$
$\frac{f(0)}{4.}$	0.1875

c)

The angle at the ends is about 11°<12°

$fd1(x):=\frac{d}{dx}(f(x))$	*Done*
$\tan^{-1}(fd1(2.))$	$^{-}10.6197$

d)

The largest downward slope is x = 0.67 and x = -0.67. (the inflection points)

The angle is 29°

$fd2(x):=\frac{d}{dx}(fd1(x))$	*Done*
$\text{zeros}(fd2(x),x)$	$\left\{\frac{-2}{3},\frac{2}{3}\right\}$
$xw:=\frac{2.}{3}$	0.666667
$\tan^{-1}\left(fd1\left(\frac{2}{3.}\right)\right)$	$^{-}29.3578$

f)

The condition for the parameter a of the function fa is: $2.37037 \le a \le 3.41333$

$fa(x):=\frac{4}{a\cdot x^2+4}-\frac{1}{4}$	*Done*
$\text{zeros}(fa(x),x)$	$\left\{\left\{\frac{2\cdot\sqrt{3}}{\sqrt{a}},\frac{1}{a}\ge 0\right\},\left\{\frac{-2\cdot\sqrt{3}}{\sqrt{a}},\frac{1}{a}\ge 0\right\}\right\}$
$fa(0)$	$\frac{3}{4}$
$\text{solve}\left(\frac{0.75}{\frac{4\cdot\sqrt{3}}{\sqrt{a}}}=\frac{1}{5},a\right)$	$a=3.41333$
$\text{solve}\left(\frac{0.75}{\frac{4\cdot\sqrt{3}}{\sqrt{a}}}=\frac{1}{6},a\right)$	$a=2.37037$

g)

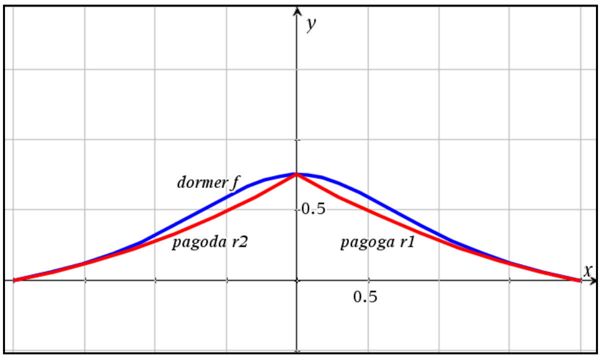
For the model of the border line of the pagoda a quadratic function is chosen.

A system of 3 linear equations to find the 3 parameters a, b, c

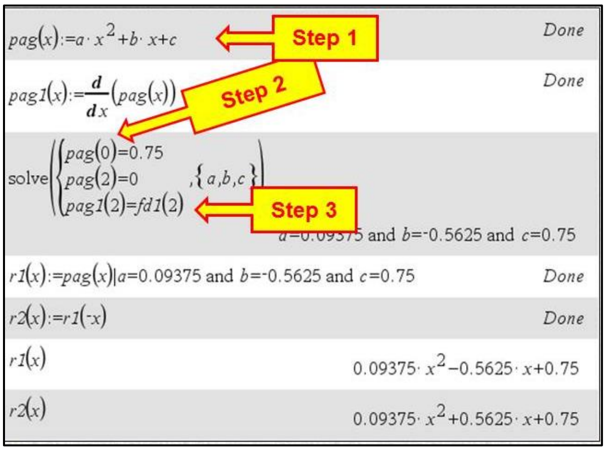
The function equations of r1 and r2

$pag(x):=a\cdot x^2+b\cdot x+c$	*Done*
$pag1(x):=\frac{d}{dx}(pag(x))$	*Done*
$\text{solve}\left(\begin{cases}pag(0)=0.75\\ pag(2)=0\\ pag1(2)=fd1(2)\end{cases},\{a,b,c\}\right)$	$a=0.09375$ and $b=^{-}0.5625$ and $c=0.75$
$r1(x):=pag(x)\vert a=0.09375$ and $b=^{-}0.5625$ and $c=0.75$	*Done*
$r2(x):=r1(-x)$	*Done*
$r1(x)$	$0.09375\cdot x^2-0.5625\cdot x+0.75$
$r2(x)$	$0.09375\cdot x^2+0.5625\cdot x+0.75$

The visualization of the given problem



Didactical comment: **A characteristic approach when using CAS**



Step 1: Developing new mathematical language elements by defining functions

Step 2: Instead of working with/in complex expressions, working with the names of expressions

Step 3: Direct translation of verbal given information into the language of mathematics

AUSTRALIA NSW
Section 2: EX.12_1
Scient. Calc

PROB | A 1 | A 2 | A 3 | A 4 | C 2

(b) In a chemical reaction, a compound X is formed from a compound Y. The mass in grams of X and Y are $x(t)$ and $y(t)$ respectively, where t is the time in seconds after the start of the chemical reaction.

Throughout the reaction the sum of the two masses is 500 g.

At any time t, the rate at which the mass of compound X is increasing is proportional to the mass of compound Y.

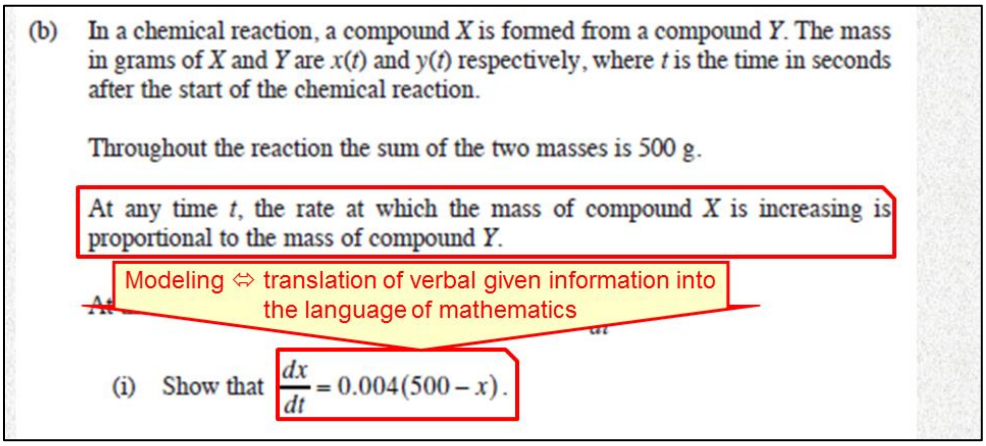
At the start of the chemical reaction, $x = 0$ and $\frac{dx}{dt} = 2$.

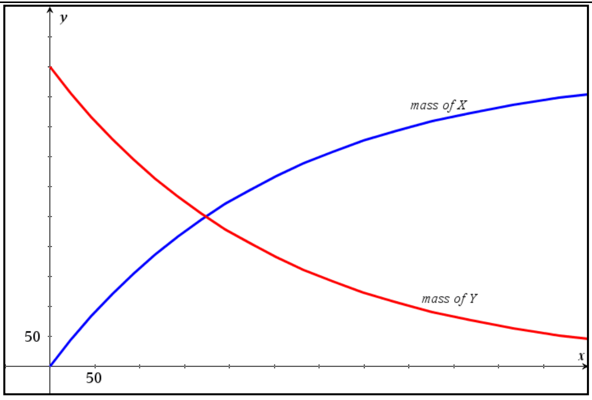
(i) Show that $\frac{dx}{dt} = 0.004(500 - x)$.

(ii) Show that $x = 500 - Ae^{-0.004t}$ satisfies the equation in part (i), and find the value of A.

Didactical comment:

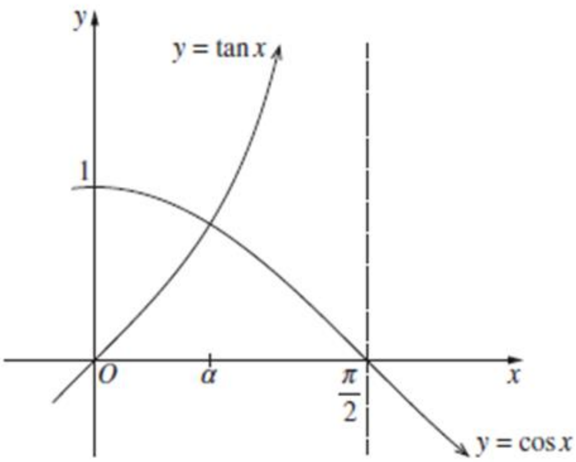
Modeling means translating the verbal given information into the langauge of mathematics



In New South Wales only scientific calculators are allowed ⇨ the expected solution is:	$\text{Precondition: If } x = 0 \quad \frac{dx}{dt} = 2$ $\frac{dx}{dt} = c \cdot (500 - x)$ $\frac{dx}{(500 - x)} = c \cdot dt$ $\int \frac{dx}{(500 - x)} = \int c \cdot dt$ $-\ln(500 - x) = 0.004 \cdot t + c1$ $500 - x = A \cdot e^{-0.004 \cdot t}$ $x = 500 - A \cdot e^{-0.004 \cdot t}$ $0 = 500 - A \cdot e^{0}$ $A = 500$
b) Calculating the constant Solving the differential equation with CAS (mode „auto“) Solving the differential equation with CAS in the „exact mode“	$x' = c \cdot (500 - x)$ → $x' = -c \cdot (x - 500)$ $2 = c \cdot 500$ → $2 = 500 \cdot c$ $c := 0.004$ → 0.004 $\text{deSolve}(x' = c \cdot (500 - x), t, x)$ → $x = c1 \cdot (0.996008)^{t} + 500.$ $\text{deSolve}(x' = c \cdot (500 - x), t, x)$ → $x = c2 \cdot e^{\frac{-t}{250}} + 500$ $\text{solve}(2 = c \cdot (500 - c2 - 500), c2)$ → $c2 = -500$ $x(t) := 500 - 500 \cdot e^{\frac{-t}{250}}$ → *Done* $y(t) := 500 - x(t)$ → *Done*
The graphs of the two functions	

AUSTRALIA NSW
Section 2: EX.12_2
Scient. Calc

(c) The graphs of $y = \tan x$ and $y = \cos x$ meet at the point where $x = \alpha$, as shown.



(i) Show that the tangents to the curves at $x = \alpha$ are perpendicular.

(ii) Use one application of Newton's method with $x_1 = 1$ to find an approximate value for α. Give your answer correct to two decimal places.

c) i)

A condition for orthogonal tangents is, that the product of the slopes is -1:

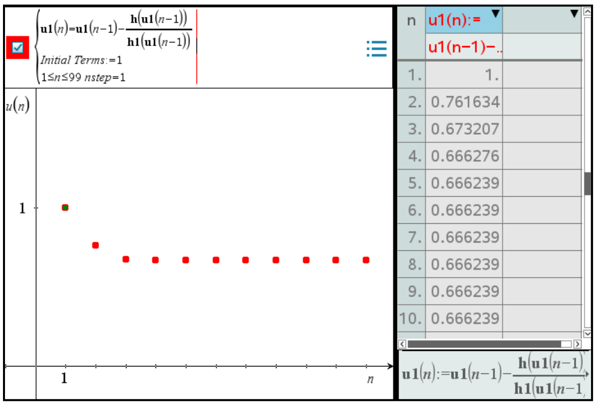
$f(x):=\tan(x)$	*Done*
$g(x):=\cos(x)$	*Done*
$f1(x):=\frac{d}{dx}(f(x))$	*Done*
$g1(x):=\frac{d}{dx}(\cos(x))$	*Done*
⚠ $f1(\alpha)\cdot g1(\alpha)=-1$	$\frac{-\sin(\alpha)}{(\cos(\alpha))^2}=-1$
$\left(\frac{-\sin(\alpha)}{(\cos(\alpha))^2}=-1\right)\cdot -\cos(\alpha)$	$\tan(\alpha)=\cos(\alpha)$

d) ii)

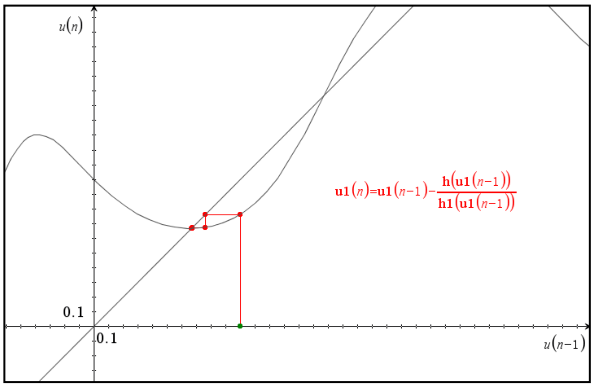
Developing a model for Newtons method by using a recursive model

$h(x):=f(x)-g(x)$	*Done*
$h1(x):=\frac{d}{dx}(h(x))$	*Done*
$u1(n):=u1(n-1)-\frac{h(u1(n-1))}{h1(u1(n-1))}$	*Done*

“Time mode”: u(n) is a function of n



“Web mode“: u(n) is a function of u(n-1)



AUSTRALIA SA Part 2 Ex. 16 Graph. Calc.

PROB | **C 2** | **A 1** | **A 2** | **A 3** | **A 4**

In drag races, cars are initially stationary on the starting line and then travel 400 metres in a straight line in as short a time as possible.

Car A (shown in the photograph) competes in drag races.

The design features of Car A are considered in the development of a mathematical model to predict the speed (*v*A in metres per second) of the car *t* seconds after it leaves the starting line.

The model assumes that all the components of Car A work optimally and that the driver attempts to finish in as short a time as possible. The model is useful for predictions only for the first 7 seconds after the car leaves the starting line.

The model is: $va = \dfrac{98}{1+19\cdot e^{-2t}} - 4.9 \quad \text{for } 0 \le t \le 7$

a) Find the maximum speed of Car A according to this model.

b) Calculate $\int_0^2 va(t)dt$ correct to the nearest whole number

Interpret your answer in the context of Car A competing in a drag race.

c) Complete the following table according to the model of Car A

t (seconds)	0	2	4	6
distance (meters) of Car A after t seconds	0			

Car B is another drag-racing car.

The model for Car B is $vB = \dfrac{85}{1+9\cdot e^{-3t}} - 8.5 \quad \text{for } 0 \le t \le 7$

d) Complete the following table according to the model of Car A

t (seconds)	0	2	4	6
distance (meters) of Car B after t seconds	0			

Unnacessary calculation competence when using CAS

e) If Car A races against Car B, which car would win the ra
Explain your answer.

f) Find $\dfrac{dy}{dt}$ if $y = \dfrac{1}{2}\cdot\ln(e^{2t}+k)$ where k is a positive real constant.

Hence show that $\int\left(\dfrac{98}{1+19\cdot e^{-2t}} - 4.9\right)dt = 49\cdot\ln(e^{2t}+19) - 4.9t + c$

Hence determine the time Car A takes to travel the 400 metres, correct to two decimal places.

a)

No zeros of the 1st derivative => no relative extremum.

va1(x)>0 for any x ⇨ va is strictly increasing. absolute maximum at x=7

Maximum speed: 93 m/s = 335 km/h

$va(x):=\frac{98}{1+19\cdot e^{-2\cdot x}}-4.9$	*Done*
$va1(x):=\frac{d}{dx}(va(x))$	*Done*
$\text{zeros}(va1(x),x)$	$\{\}$
$va1(x)$	$\frac{3724.\cdot e^{2\cdot x}}{(e^{2\cdot x}+19.)^2}$
$va(7)$	93.0985

b)

The distance travelled by Car A after x seconds.

The distance after 2 seconds is 54 m.

$sa(x):=\int_0^x va(t)\,dt$	*Done*
$sa(2)$	54.0415

The distance-function sB of car B

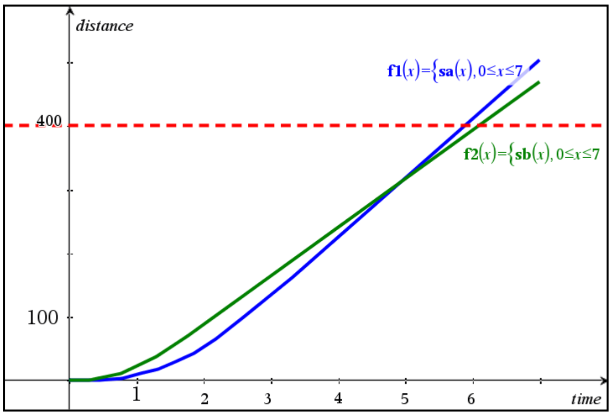
$vb(x):=\frac{85}{1+9\cdot e^{-3\cdot x}}-8.5$	*Done*
$sb(x):=\int_0^x vb(t)\,dt$	*Done*

e)

Car A needs 5.87 s, Car B needs 6.08 s ⇨ Car A wins the race

⚠ $\text{solve}(400=sa(x),x)$	$x=-82.1456$ or $x=5.87308$
⚠ $\text{solve}(400=sb(x),x)$	$x=-47.41$ or $x=6.08157$

An alternative approach: The graphical solution



Denmark European Exam Part 2 Example 1 CAS

CALC | A 2 | A 4 | C 4

Given is fa with a>0

$$fa(x)=\begin{cases}\dfrac{x^2}{2a}-2x+\dfrac{3}{2}a\,, & x<a\\[2ex] (x-a)e^{2-\frac{x}{a}}, & x\ge a\end{cases}$$

For x=1 th function is called f1

Look for:

a) the graph of f1 and the intersection points with the axes
b) Asymptotes of f1
c) Extrema
d) Points of inflection
e) Calculate $\int_1^{\infty} f1(x)\,dx$ and give a geometric interpretation
f) Show that fa is continous at x=a
g) Show that functions fa have a maximum for any x>a and show that the maxima are situated on a straight line
h) Find the equation of a tangent with respect to a at the intersection point with the y-axes and show that all these tangents are parallel

a)

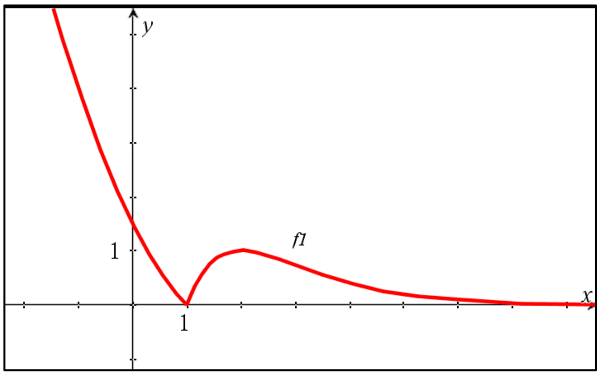
Defining the functions

$$fa(x):=\begin{cases}\dfrac{x^2}{2\cdot a}-2\cdot x+\dfrac{3}{2}\cdot a, & x<a\\[2ex] (x-a)\cdot e^{2-\frac{x}{a}}, & x\ge a\end{cases}$$ *Done*

$f1(x):=fa(x)|a=1$ *Done*

$f1(x)$ $\quad\begin{cases}\dfrac{x^2}{2}-2\cdot x+\dfrac{3}{2}, & x<1\\[2ex] (x-1)\cdot e^{2-x}, & x\ge 1\end{cases}$

The graph of f1 and the intersection points with the axes



$\text{zeros}(f1(x),x)$ $\quad\{1\}$

$f1(0.)$ $\quad 1.5$

b) c) d) e)
Asymptotes, extrema, points of inflection and integral:

$\lim_{x \to \infty} (f1(x))$	0
$der1f1(x) := \frac{d}{dx}(f1(x))$	*Done*
$der2f1(x) := \frac{d}{dx}(der1f1(x))$	*Done*
$\text{zeros}(der1f1(x), x)$	$\{2\}$
$der2f1(2)$	-1
$\text{zeros}(der2f1(x), x)$	$\{3\}$
$\int_1^{\infty} f1(x)\, dx$	$\mathbf{e}$

f)
fa(x) is continuous at x=a

⚠ $fa(a)$	0
$\lim_{h \to 0} (fa(a-h))$	0
$\lim_{h \to 0} (fa(a+h))$	0

g)
For a>0 the functions fa have a maximum. All maxima are situated on the straight line $y = \frac{1}{2} \cdot x$

$fa2(x) := (x-a) \cdot \mathbf{e}^{2-\frac{x}{a}}$	*Done*
$der1fa2(x) := \frac{d}{dx}(fa2(x))$	*Done*
$der2fa2(x) := \frac{d}{dx}(der1fa2(x))$	*Done*
⚠ $\text{zeros}(der1fa2(x), x)$	$\{2 \cdot a\}$
⚠ $der2fa2(2 \cdot a)$	$\frac{-1}{a}$
⚠ $fa2(2 \cdot a)$	a

h)
All the equations of the tangents of fa at the intersection points with the y-axes have the slope -2 ⇨ they are parallel

$fa1(x) := \frac{x^2}{2 \cdot a} - 2 \cdot x + \frac{3}{2} \cdot a$	*Done*
$der1fa1(x) := \frac{d}{dx}(fa1(x))$	*Done*
⚠ $der1fa1(0)$	-2
⚠ $fa1(0)$	$\frac{3 \cdot a}{2}$
$t(x) := -2 \cdot x + \frac{3 \cdot a}{2}$	*Done*

Didacical comment:

The higher complexity of expressions caused by the use of CAS in such examples are not a better contribution to the goals of mathematics education. Students do not work with these expressions, they use the names of the expressions. Therefore their cognitive activities are the same as in tasks with less complex terms. When using CAS so called “curve discussions” are senseless because they could also be solved by using prepared applets which stdents develop in the “notes application”.

More complex expressions are only senseful to describe a more complex reality in applied problems.

NORWAYAY
Ex 4 Part 2
CAS

PROB | A 1 | A 2 | A 3 | C 4

Type 2

The spread of rumor:

In a town with 1200 residents a rumor is spread . Y is the number of residents who have heared about the rumor. t is the time in days after planting the rumor.

Assumption: The velocity of the spread of the rumor is at any time proportional to the product of the number of residents who know the rumor and who don't know the rumor .The proportional coefficient c = 0.0006

At t = 0 one person kmows the rumor.

a) Look for a differential equation which describes this situation.

b) Look for the time when the half of the town knows the rumor

Solving the differential equation with CAS in the Calculation mode: „*Auto*"

$y'=c\cdot y\cdot(1200-y)$ and $c=6.\text{E-}4$ $\quad\quad y'=-0.0006\cdot y\cdot(y-1200)$ and $c=0.0006$

deSolve$(y'=-6.\text{E-}4\cdot y\cdot(y-1200)$ and $y(0)=1,x,y)$ $\quad\quad y=\dfrac{1200.\cdot(2.05443)^x}{(2.05443)^x+1199.}$

$f(x):=\dfrac{1200.\cdot(2.05443)^x}{(2.05443)^x+1199.}$ $\quad\quad$ *Done*

solve$(f(x)=600,x)$ $\quad\quad x=9.84619$

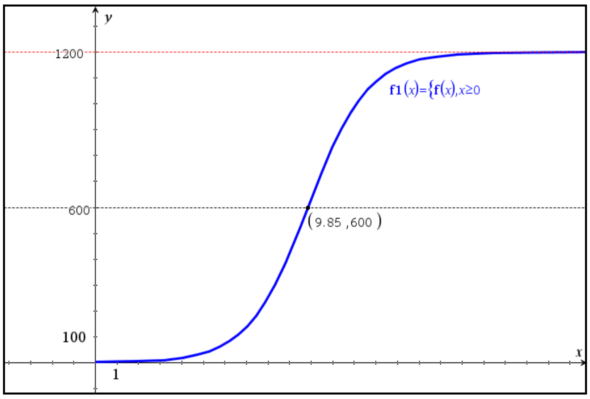
Solving the differential equation with CAS in the Calculation mode: „*Exact* "

deSolve$(y'=-6.\text{E-}4\cdot y\cdot(y-1200)$ and $y(0)=1,x,y)$ $\quad\quad y=\dfrac{1200\cdot e^{\frac{18\cdot x}{25}}}{e^{\frac{18\cdot x}{25}}+1199}$

$fexact(x):=\dfrac{1200\cdot e^{\frac{18\cdot x}{25}}}{e^{\frac{18\cdot x}{25}}+1199}$ $\quad\quad$ *Done*

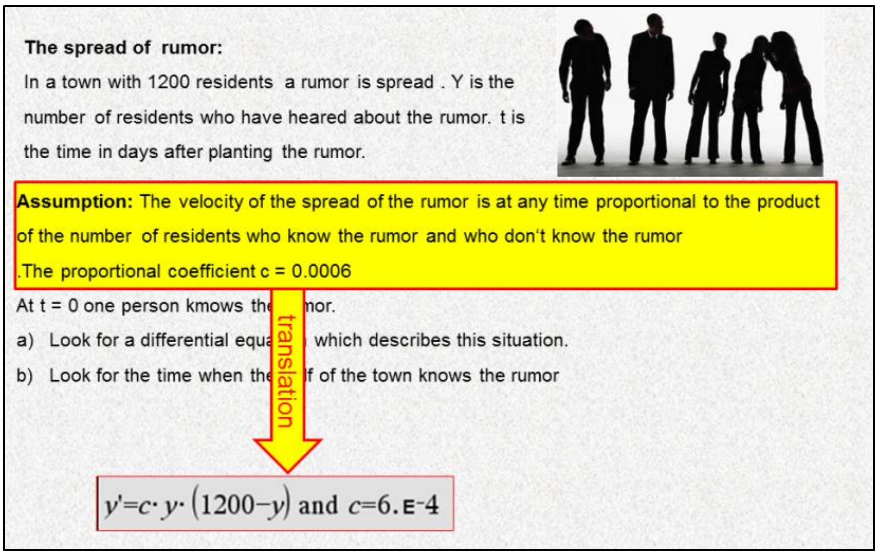
solve$(fexact(x)=600,x)$ $\quad\quad x=\dfrac{25\cdot\ln(1199)}{18}$

For the interpretation of the growth process the graph is useful and a graphical solution of task (b) is also possible



Didactical comment:

In this example can also be seen that “modeling” means to translate the verbal given information into the language of mathematics:



NORWAYAY Ex 3 Part 2 CAS

PROB | **A 1** | **A 2** | **A 3** | **C 4**

Type 2

Given is the function f with $f(x) = x^2$

A(a,a2) and B(b,b2)) with a<b are lie on the graph.

a) Calculate the area T which is limited by the graph f and the line segment AB (see figure 1.)

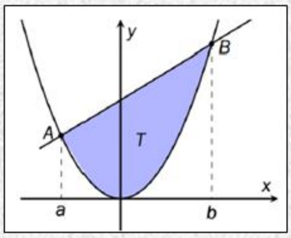


Figure 1.

b) Calculate the area S of the triangle ABC (see figure 2.)

Show that $S = \frac{1}{8}\cdot(b-a)^3$

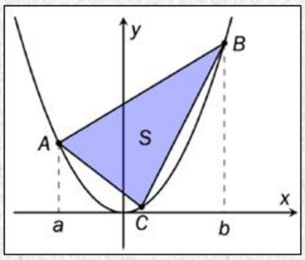
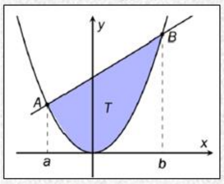


Figure 2.

c) Look for the proportion $\frac{T}{S}$

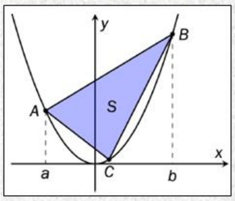
Task (a):



$$area := \frac{b^2+a^2}{2}\cdot(-a+b) - \int_a^b f(x)\,dx \quad \rightarrow \quad \frac{-a^3}{6} + \frac{a^2\cdot b}{2} - \frac{a\cdot b^2}{2} + \frac{b^3}{6}$$

$$\text{factor}\left(\frac{-a^3}{6} + \frac{a^2\cdot b}{2} - \frac{a\cdot b^2}{2} + \frac{b^3}{6}\right) \quad \rightarrow \quad \frac{-(a-b)^3}{6}$$

Task (b):



$$area1 := \frac{b^2+a^2}{2}\cdot(-a+b) - \frac{a^2+\left(\frac{a+b}{2}\right)^2}{2}\cdot\left(-a+\frac{a+b}{2}\right) - \frac{b^2+\left(\frac{a+b}{2}\right)^2}{2}\cdot\left(b-\frac{a+b}{2}\right) \quad \rightarrow \quad \frac{-(a-b)\cdot\left(a^2-2\cdot a\cdot b+b^2\right)}{8}$$

$$\text{factor}\left(\frac{-(a-b)\cdot\left(a^2-2\cdot a\cdot b+b^2\right)}{8}\right) \quad \rightarrow \quad \frac{-(a-b)^3}{8}$$

$$\frac{area}{area1} \quad \rightarrow \quad \frac{4}{3}$$

Especially when using CAS the **competence of recognizing structures** is very important

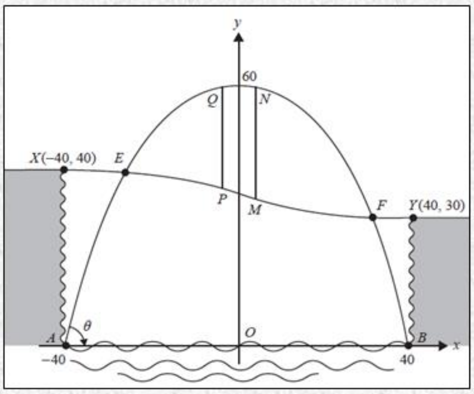
Didactical comment:

Not only in application oriented examples but also in pure mathematical problems modeling is an important and useful contribution to the goals of mathematics education.

Australia Victoria Part 2 Example 2 CAS	PROB	C 4	A 1	A 2	A 3	A 4

A city is located on a river that runs through a gorge. The gorge is 80 m across, 40 m high on one side and 30 m high on the other side.
A bridge is to be built that crosses the river and the gorge. A diagram for the design of the bridge is shown below.

The road from *X* to *Y* across the gorge has gradient zero at *X* (– 40, 40) and at *Y* (40, 30), and has equation $y = \frac{x^3}{25600} - \frac{3x}{16} + 35$



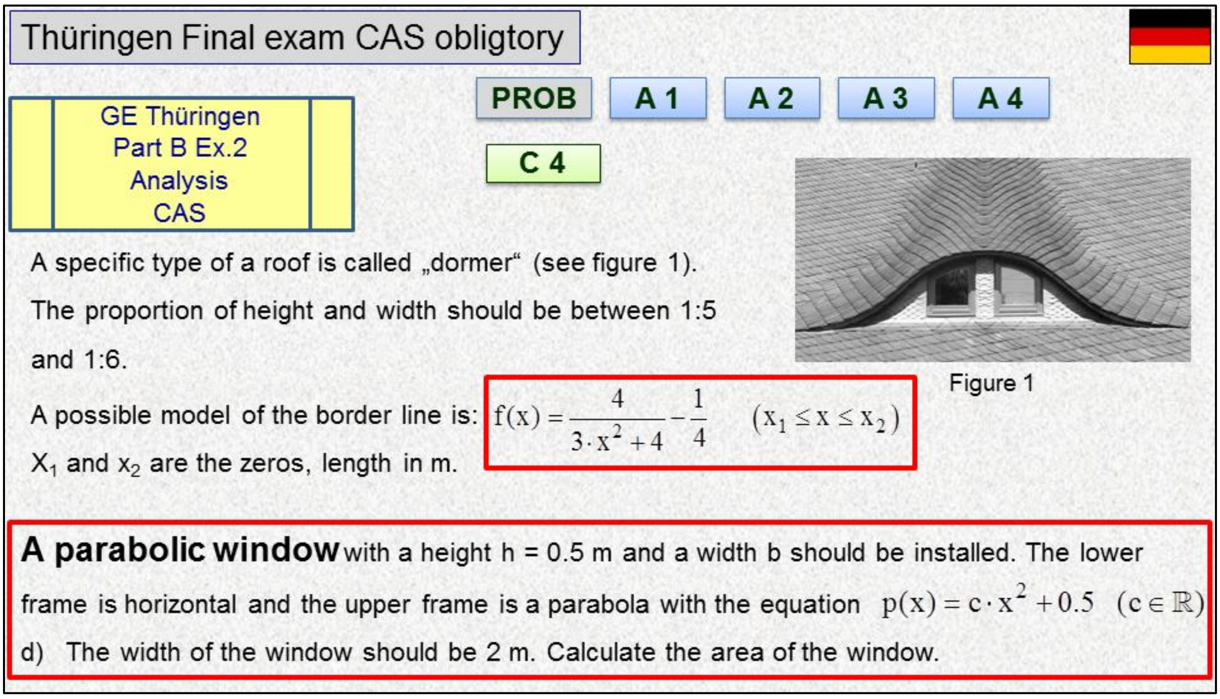
The original version of this Australian example was classified as an example of type T1 because the mathematical models are given.

But it could also be an example of type T2 if the information about the road across the gorge is used to find a polynomial function with grade 3 as a mathematical model:

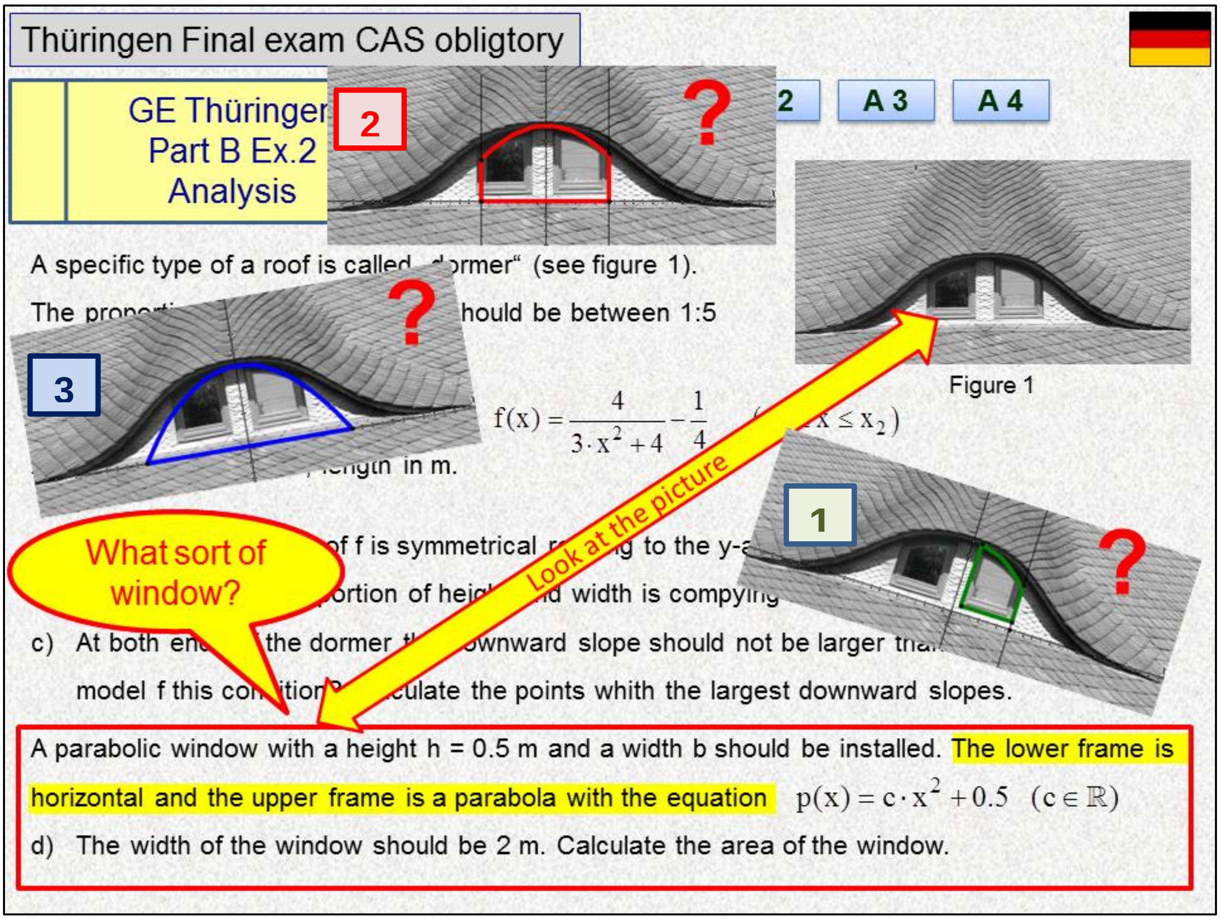
Modeling: Find the equation of the road (a polynomial function with grade 3) which has the gradient zero at X(-40, 40) and at Y(40, 30)

CAS	
$r(x):=a\cdot x^3+b\cdot x^2+c\cdot x+d$ *Done* $r1(x):=\frac{d}{dx}(r(x))$ *Done* $\text{solve}\left(\begin{cases} r(-40)=40 \\ r(40)=30 \\ r1(-40)=0 \\ r1(40)=0 \end{cases},\{a,b,c,d\}\right)$ $a=\frac{1}{25600}$ and $b=0$ and $c=\frac{-3}{16}$ and $d=35$ $road(x):=r(x)\vert a=\frac{1}{25600}$ and $b=0$ and $c=\frac{-3}{16}$ and $d=35$ *Done* $road(x)$ $\frac{x^3}{25600}-\frac{3\cdot x}{16}+35$	Step 1: Developing new mathematical language elements by defining functions Step 2: Instead of working with/in complex expressions, working with the names of expressions Step 3: Direct translation of verbal given information into the language of mathematics

An example which shows that the application oriented question is more difficult than the mathematical problem:



In my opinion this information about the parabolic window is not unique:



Confusing is also that the given picture shows a window which is not correspond with the expected solution. The expected solution is version 3.

Literature

Böhm, J., a.o., 2004: „The Case for CAS“. T3 Europe (Teachers Teaching with Technology). ISBN 3-934064-45-0

Buchberger, B., 2002: (Research Institute for Symbolic Computation, University of Linz, Austria): “Logic, Mathematics, Computer Science: Interactions” Talk at LMCS 2002. Castle of Hagenberg, October 2002.

Dörfler, W., 1991: „Der Computer als kognitives Werkzeug und kognitives Medium“. In Computer-Mensch-Mathematik, Schriftenreihe Didaktik der Mathematik, Universität Klagenfurt, Band 21. ISBN 3-519-02799-2.

Heugl, H., 2014: „Mathematikunterricht mit Technologie – ein didaktisches Handbuch mit einer Vielzahl von Aufgaben“. Veritas-Verlag, Linz. ISBN 978-3-7101-0431-2

NCTM: „Principles and Standards for School Mathematics“. http://www.nctm.org/Standards-and-Positions/Principles-and-Standards/